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CMS HISTOGRAM, DENSITY ESTIMATION AND PROBABILITY PLOTTING ROUTINES, WITH AN APPLICATION TO THE ANALYSIS OF THE OUTPUT OF A SIMULATION OF A CORRELATED QUEUE.

Georgios Ioannis Danicas

NAVAL POSTGRADUATE SCHOOL

Monterey, California

THESIS

CMS Histogram, Density Estimation and Probability Plotting Routines, with an Application to the Analysis of the Output of a Simulation of a Correlated Queue

by

Georgios Ioannis Danikas

December 1977

Thesis Advisor: P. A. W. Lewis

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The object of this thesis has been twofold. The first object was to develop FORTRAN versions of several existing APL programs which were designed to analyze univariate data. In particular the programs were designed to test for exponentiality and normality of the data and, by sectioning or jacknifing, obtain estimates of sampling variances of sample moments. The

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December 1977

ABSTRACT

The object of this thesis has been twofold. The first object was to develop FORTRAN versions of several existing APL programs which were designed to analyze univariate data. In particular the programs were designed to test for exponentiality and normality of the data and, by sectioning or jackknifing, obtain estimates of sampling variances of sample moments. The second object of the thesis was to use these programs in a simulation study of first-come first-served queues in which the service times and the inter-arrival times are exponentially distributed but dependent. The dependence is introduced by using the mixed moving average autoregresive structure (EARMA(p,q)) for exponential sequences introduced by Lewis and co-workers. Four models of correlated queues are introduced, giving autocorrelated and cross-correlated service and arrival times in various degrees. The simulation study gives a quantitative idea of the effect of correlation on the mean waiting time and the distribution of the waiting time.

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I. INTRODUCTION

The object of this thesis has been twofold.

The first object was to develop FORTRAN versions of several existing APL programs which were designed to analyze univariate data, in particular to test for exponentiality and normality of the data and, by sectioning or jackknifing obtain estimates of variances of sample moments.

The second object was to use these programs in a simulation study of first-come first-served queues in which the service times and the inter-arrival times are exponentially distributed but dependent. The dependence is introduced by using the mixed moving average autoregressive structure (EARMA(p,q)) for exponential sequences introduced by Gaver and Lewis (1977), Lawrance and Lewis (1977), Jacobs and Lewis (1977) and Lawrence and Lewis (1978). Four schemes are introduced which give autocorrelated and cross-correlated service and arrival times in various degrees.

Since no analytical properties of these queues can be derived, their properties are studied by simulation. Because the EARMA queues are neither regenerative nor Markovian there are several problems in the simulation study of knowing when one can assume that a steady state has been reached in the simulation and of obtaining confidence intervals for estimated parameters. This problem is handled by generating a number of independent sample paths and comparing statistics obtained from the replications at several points along the sample paths. Only the waiting

time process (W_N) is studied in this thesis, not the state space process N(t).

Unfortunately the simulation study of the EARMA-type queue was hampered by the size and speed of the IBM System 360/67 system at the Naval Postgraduate School. Consequently the results were not as extensive as had been hoped for. Fairly detailed results are given for the case where the service time process is autoregressive; one case where the traffic intensity is 0.99 and the correlation parameter 0.98, was intensively investigated. This gives some idea of the length of the transient in the queue and of the inflation of the mean waiting time because of the correlation in the service process. Some investigation of the case where the service times and inter-arrival times are cross-correlated are given.

In section II we describe the basic histogram and statistics computation package 'HISTGS/F' and apply it to two sets of telephone error data. In section III the plotting method and the plotting Subroutines 'NORMPL', 'EXPLT' are described and we test them with various generated data. In section IV we describe the assessment of variability Subroutines 'SECTN', 'JACK' and we apply them to telephone data set 1. In section V the Subroutine 'LIST' is described and we apply it to two sets of telephone error data. In section VI we describe and analyze the EARMA(p,q) model as well the program which we developed in order to simulate it. Computer program listings are provided after section VI.

II. SUBROUTINE HISTGS/HISTFS

A. DESCRIPTION

Here the Subtoutine 'HISTGS/HISTFS' is presented. 'HISTGS/HISTFS' is used for estimating the probability density function from a given set of data and at the same time computing some basic statistics. Basically this Subroutine is the library FORTRAN-Subroutine HISTF/G, which was created at N.P.S. by D. W. Robinson. It has been modified by adding the new entry point 'LIMITS'.

The new entry point may be used for wild data or for data having a mixed distribution. Therefore it is a useful tool for that user involved with this kind of data. Simply, the user has to define those values in the range of data (sectioning the data into several disjoint sections) that he/she believes are useful cut points. For each section of data the user may have the probability density function as well as the basic statistics (as HISTF/G does). Also by sectioning the data over its range, the user may have a histogram of the random variable when it is conditioned to be between given limits.

The number of sections that the entry point 'LIMITS' can accept is as many as 50.

A complete description of how 'HISTGS/HISTFS' operates is given in the subroutine. On the other hand, a summary of the subroutine is given by typing on the terminal the

command DESCRIBE HISTGS under CMS (Cambridge Monitor System, Ref. 4). Typing 'DESCRIBE HISTGS' the following response will be given on the terminal:

SUBROUTINE HISTGS/HISTFS

'HISTGS/HISTFS' gives you a histogram of a set of data along with the estimate of a set of basic statistics. Also it gives you the following options:

1. Displaying a smoothed empirical probability density function over the histogram,
2. Sectioning the data into several disjoint sections (no more than 50) and then having a histogram and the estimate of basic statistics for each section,
3. Scaling the histogram,
4. Displaying just the histogram (with or without the density function) and no statistics.

The estimated basic statistics are: Mean, Median, Trimean, Midrange, Geometric Mean, Harmonic Mean, Variance, Standard Deviation, Coefficient of Variation, Mean Deviation, Range, Midspread, Third and Forth Central Moment, Skewness, Kurtosis, Beta1, Beta2, Maximum, Minimum, and Quantiles.

CALLING SEQUENCES

```
CALL   HISTGS  ( X, N, NBAR )
CALL   HISTFS  ( X, N, NBAR )
CALL   LIMITS  ( X, N, XLIM, LIMS )
CALL   NOSTAS
CALL   STATS
CALL   FIXS  ( SCALE )
CALL   NOFIXS
CALL   PRCMAX  ( PSC )
CALL   NOPRMX
```

ARGUMENTS

X	Array of data values
N	Number of data values
NEAR	Number of bars in the histogram
XLIM	Array of cut points
LIMS	Number of cut points
SCALE	Vector of two values to scale the data
PSC	A real variable (between 0. and 1.) defining the maximum value of the probability axis.

If the entry point NOSTAS is called before calling any of the entry points HISTFS, HISTGS, or LIMITS then no statistics are printed, otherwise the statistics are printed by default.

By calling the entry point FIXS the histogram scale may be fixed and remains set unless it is reset by another call to FIXS or allowed to float (the default) by calling NOFIXS. The reason for fixing the scales is so that when comparing more than one batch of data, we can get comparably scaled histograms.

By calling PROMAX, the maximum value of the probability axis (Y) can be set to PSC, which will remain in effect until NOPRMX is called (the default).

More information is given in the subroutine.

B. USING 'HISTGS/HISTFS' WITH TELEPHONE DATA 1 AND TELEPHONE DATA 2

Here 'HISTGS/HISTFS' has been used to plot the set of Telephone Data 1 and Telephone Data 2. The algorithmic

procedure for both sets of data is the same. That is, the entry point 'HISTFS' is called first and then the entry point 'LIMITS'. Four cut points have been used (1, 2, 141, 86000) sectioning the data into three disjoint sections as follows:

Section 1 : Consists of data points $X_{(i)}$, such that $1 \leq X_{(i)} < 2$.

Section 2 : Includes the data points $X_{(i)}$, such that $2 \leq X_{(i)} < 141$.

Section 3 : Includes the data points $X_{(i)}$, such that $142 \leq X_{(i)} < 86000$.

No print-out for section 1 is given because of the constant value of the data points.

Figures 1 through 6 give the histogram and the basic statistics for both sets of data (For the entire set and for sections 2 and 3).

We observe that both of data sets are so unruly that the use of the entry point 'LIMITS' of 'HISTGS' is suggested.

Observing Figures 1 and 4 we may conclude that both data sets appear to be the same, having some geometric or exponential distribution. But as we will see later this is not true. Thus the histogram, in that case, which is obtained by using just the entry point 'HISTFS' is not helpful and leads us to a wrong conclusion.

'LIMITS' is applied now to give us a useful answer :

Observing figures 2 and 5, obtained by using the entry point 'LIMITS', we may conclude that the data sets do not have the same distribution (at least in that interval). In addition data set 2 is far from having a geometric or an

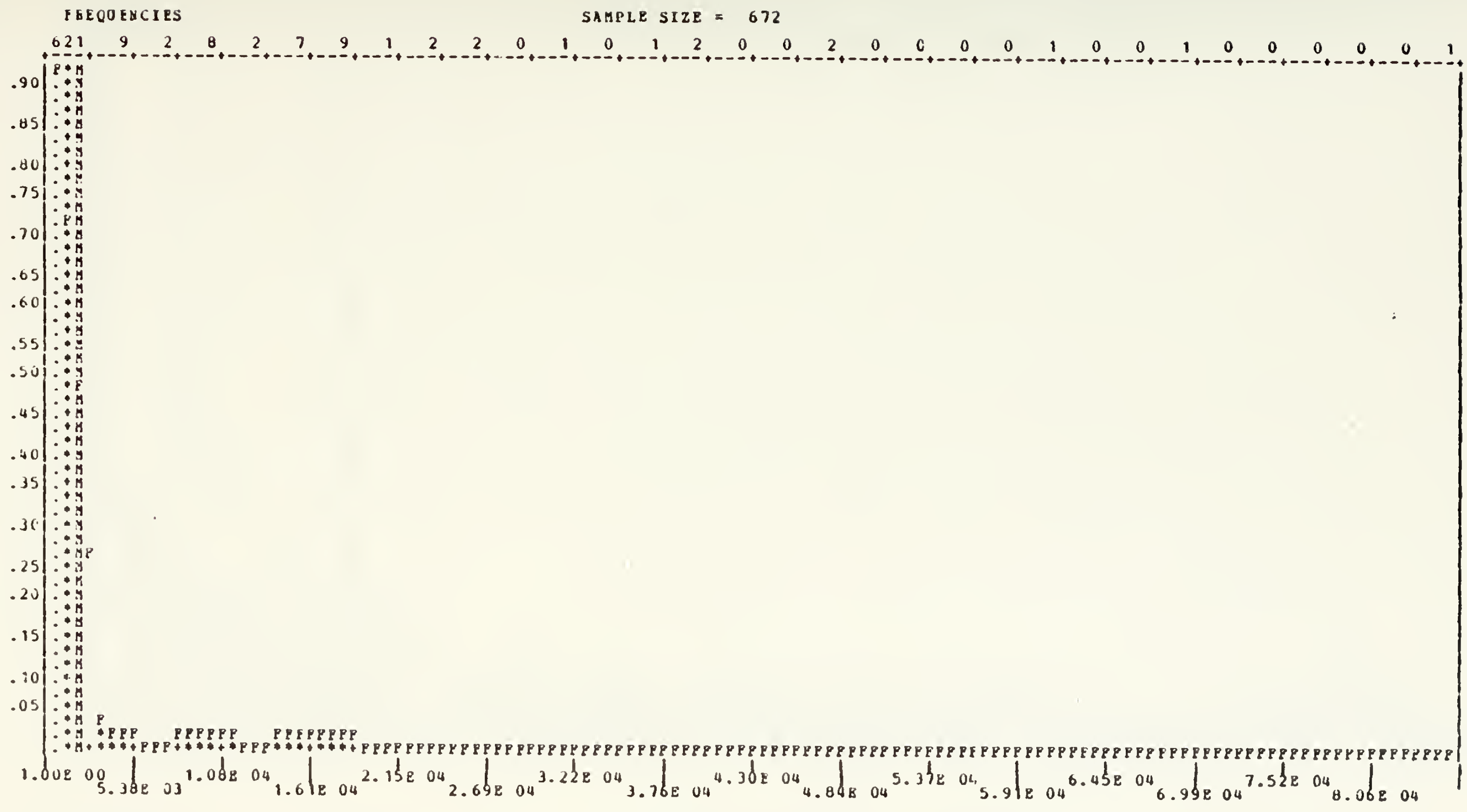
exponential distribution, as it can be seen by the histogram of the entire set of data. This is a conclusion which cannot be obtained by without use of the entry point 'LIMITS'.

Observing also Figures 3 and 6 , where the histogram of the 3rd section from each data set is presented, we may conclude that this section of both data sets may have an exponential distribution, but not the same one.

The most important facet of the data which appears is the modes at about 124. The mode is much more apparent in Telephone Data 2. In fact closer investigation reveals that this mode was due to leakage of dial pulses; this leakage caused bit errors at multiples of 124 bits.

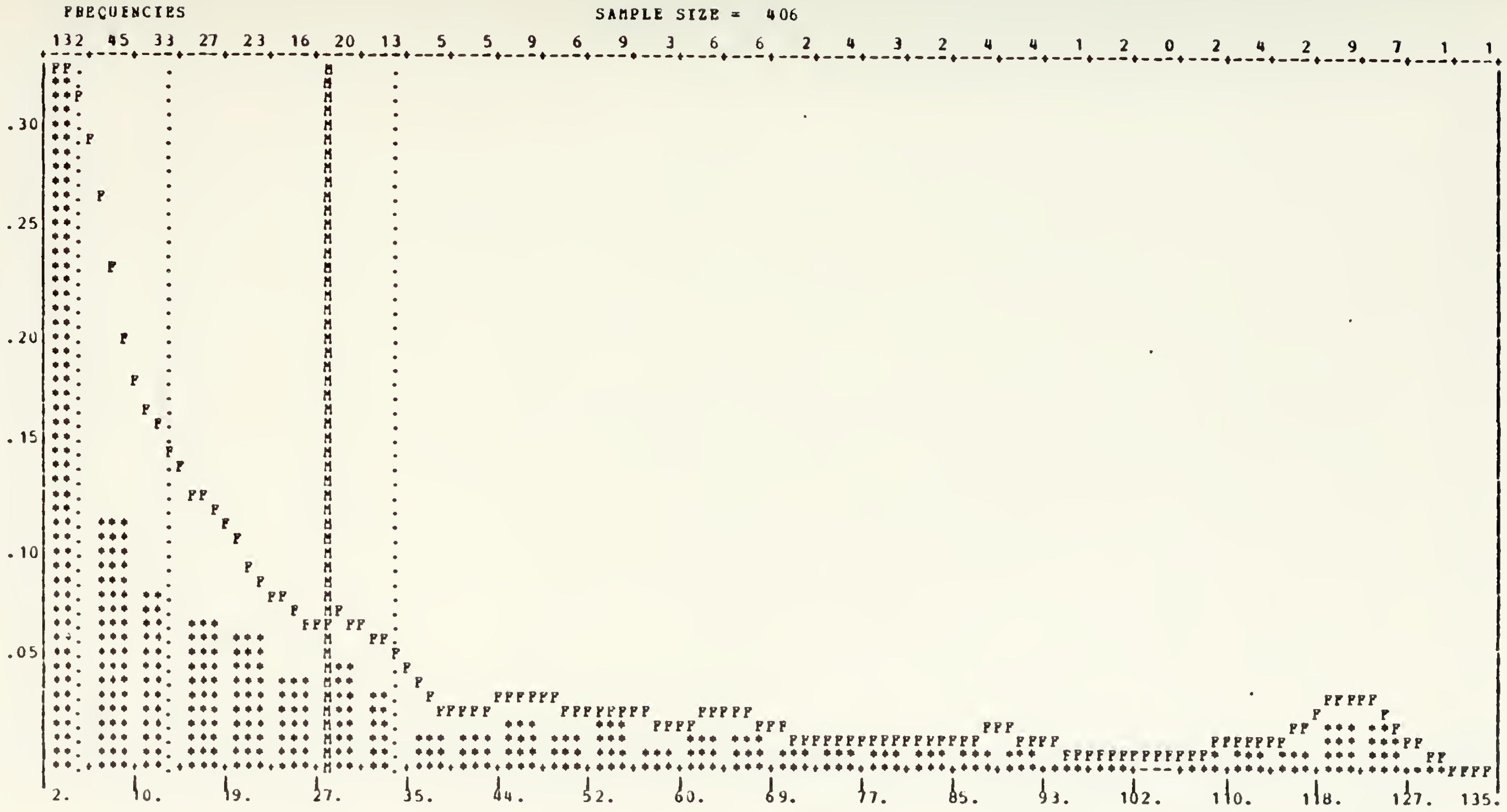
The above is a quick and informal analysis of the Telephone Data Sets obtained by using the entry point 'LIMITS' of 'HISTGS/HISTFS', and the conclusion is that both of data sets are so unruly that any formal characterization of their distribution is not so easily obtained. In addition it is apparent that a mixed distribution governs both sets of data.

Having quickly analyzed both of the telephone data sets, the use of entry point 'LIMITS' of the Subroutine 'HISTGS/HISTFS' is obvious.



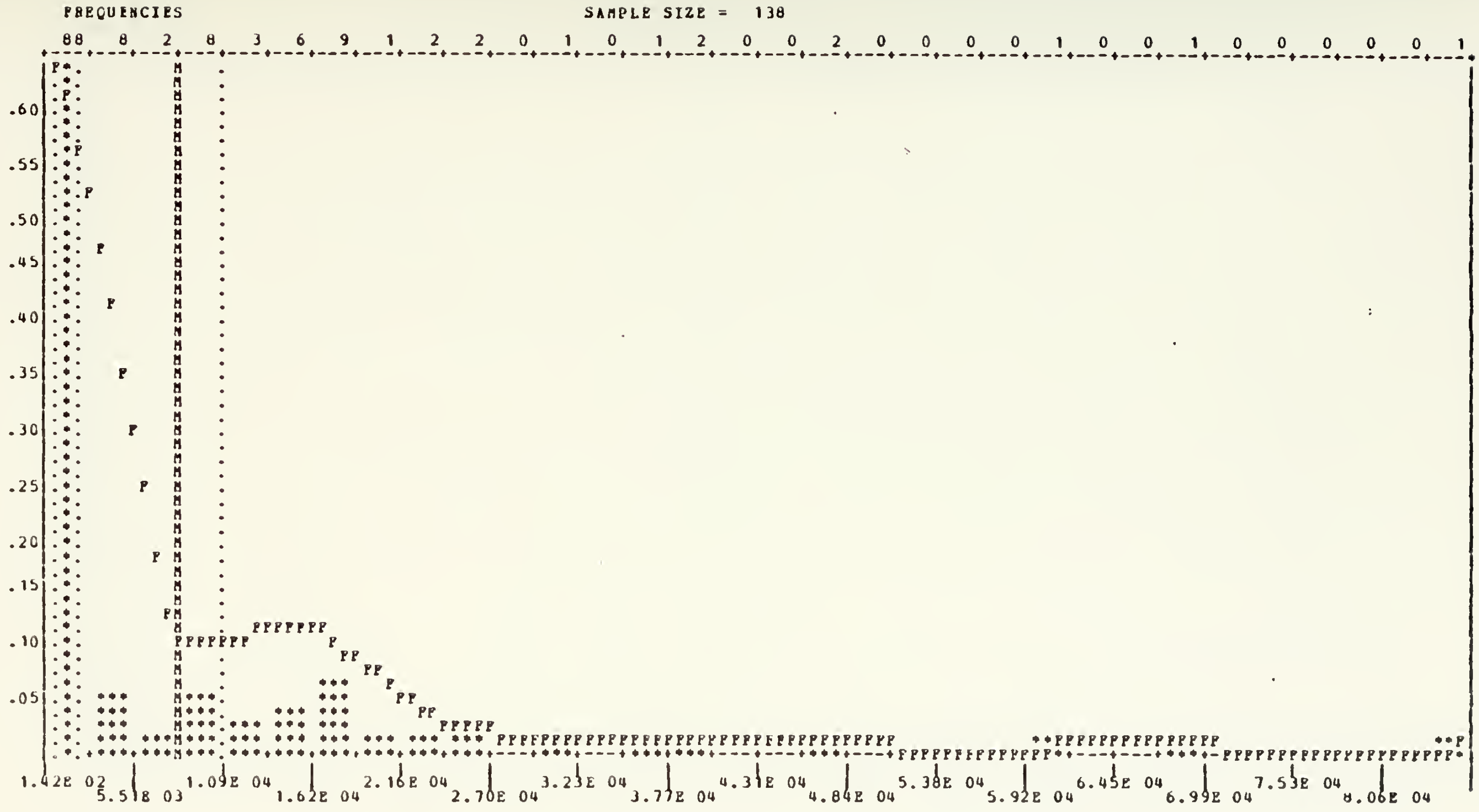
```
CENTRAL TENDENCY                  SPREAD                           HIGHER CENTRAL MOMENTS            DISTRIBUTION

MEAN         1.548219E 03         VARIANCE    4.836181E 07         M3          2.405744E 12          MINIMUM                         1.000000E 00
MEDIAN       1.400000E 01         STD DEV     6.954266E 03         M4          1.534494E 17          .10 QUANTILE                    1.000000E 00
TRIMEAN      3.050000E 01         COEF VAR    4.491784E 00         SKEWNESS    7.153120E 00          .25 QUANTILE  (HINGE)           2.000000E 00
MIDMEAN      2.219940E 01         MEAN DEV    1.544299E 03         KURTOSIS    6.260847E 01          .50 QUANTILE  (MEDIAN)          1.400000E 01
MIDRANGE     4.299700E 04         RANGE       8.599200E 04         BETA1       2.395016E 12          .75 QUANTILE  (HINGE)           9.200000E 01
GEOM MEAN    2.137872E 01         MIDSPREAD   9.000000E 01         BETA2       1.525567E 17          .90 QUANTILE                    1.270000E 03
HARM MEAN    3.547554E 00                                                                            MAXIMUM                         8.599300E 04
```

Figure 1 - TELEPHONE DATA 1. THE ENTIRE SET OF DATA.



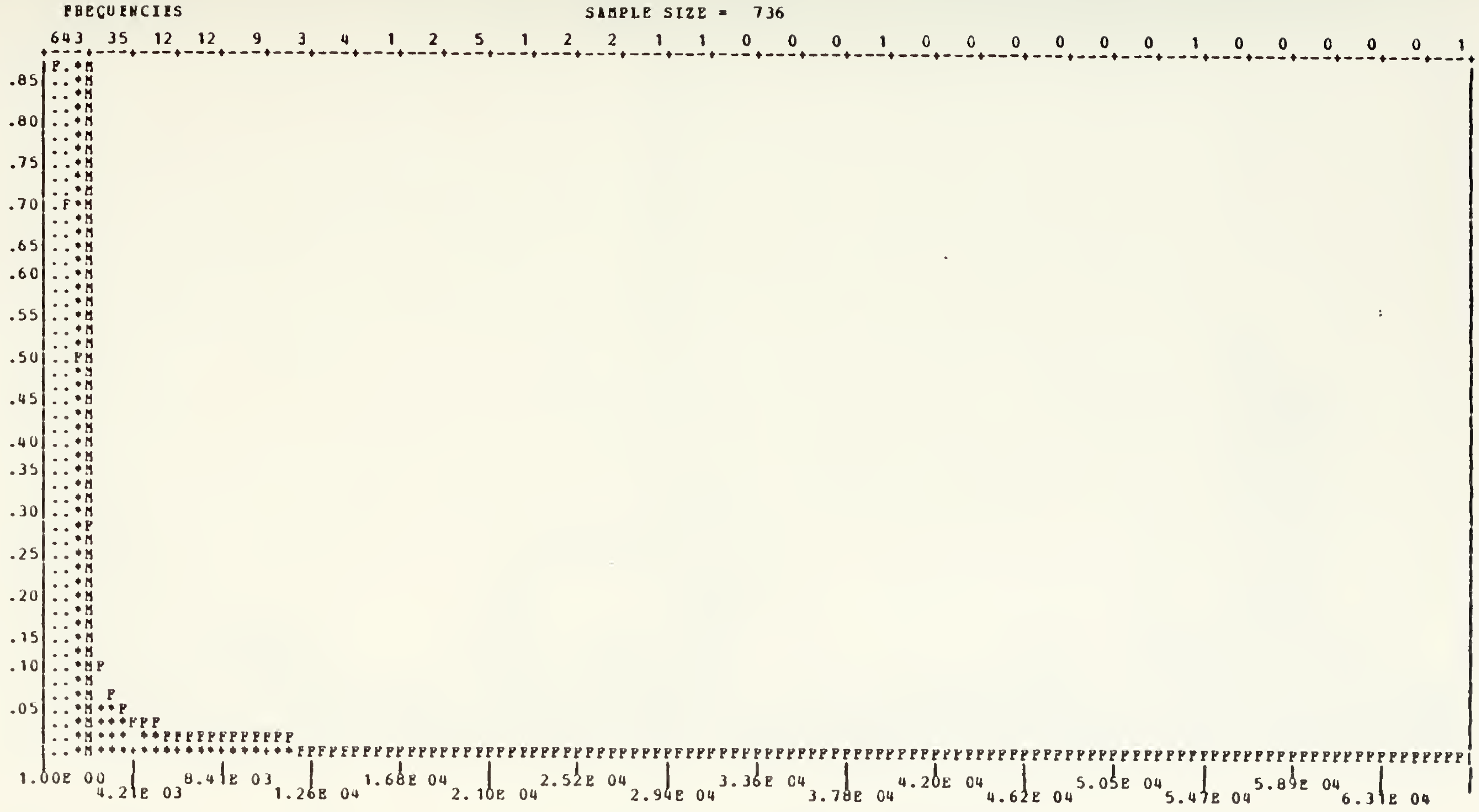
```
CENTRAL TENDENCY              SPREAD                           HIGHER CENTRAL MOMENTS       DISTRIBUTION

MEAN        2.716994E 01      VARIANCE    1.070057E 03         M3          5.920430E 04     MINIMUM                          2.000000E 00
MEDIAN      1.300000E 01      STD DEV     3.271173E 01         M4          5.680038E 06     .10 QUANTILE                     2.000000E 00
TRIMEAN     1.600000E 01      COEF VAR    1.203967E 00         SKEWNESS    1.691385E 00     .25 QUANTILE  (HINGE)            4.000000E 00
MIDMEAN     1.533823E 01      MEAN DEV    2.165269E 01         KURTOSIS    1.960636E 00     .50 QUANTILE  (MEDIAN)           1.300000E 01
MIDRANGE    6.850000E 01      RANGE       1.330000E 02         BETA1       5.876760E 04     .75 QUANTILE  (HINGE)            3.400000E 01
GEOM MEAN   1.319363E 01      MIDSPREAD   3.000000E 01         BETA2       5.640936E 06     .90 QUANTILE                     7.900000E 01
HARM MEAN   6.639325E 00                                                                    MAXIMUM                          1.350000E 02
```

Figure 2 - TELEPHONE DATA 1. 2nd SECTION.



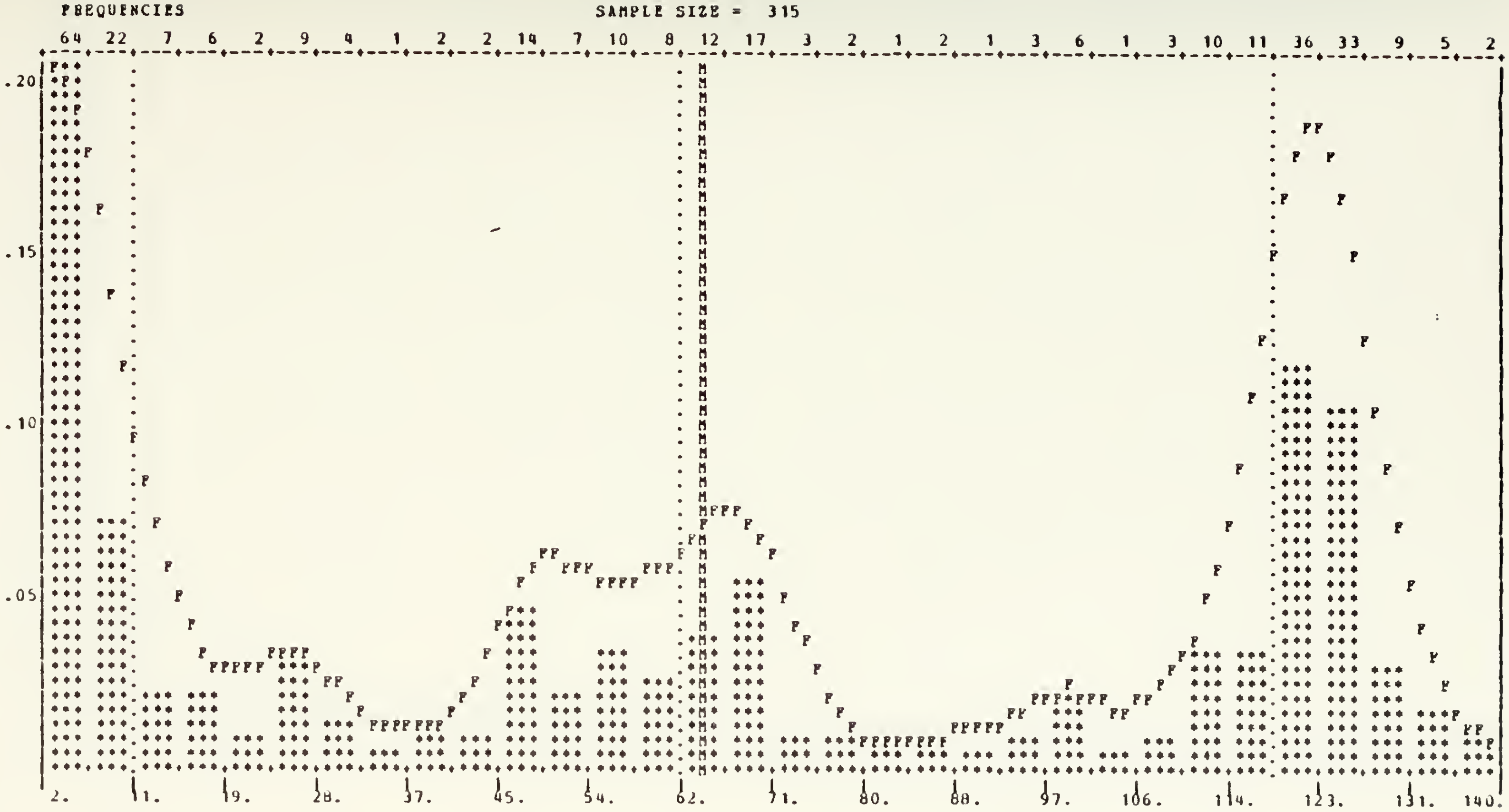
```
CENTRAL TENDENCY             SPREAD                       HIGHER CENTRAL MOMENTS      DISTRIBUTION

MEAN        7.458289E 03     VARIANCE    1.925870E 08     M3          8.258870E 12    MINIMUM                       1.420000E 02
MEDIAN      1.210000E 03     STD DEV     1.387757E 04     M4          5.241542E 17    .10 QUANTILE                  1.870000E 02
TRIMEAN     3.205500E 03     COEF VAR    1.860691E 00     SKEWNESS    3.090153E 00    .25 QUANTILE  (HINGE)         2.480000E 02
MIDMEAN     2.099600E 03     MEAN DEV    7.111348E 03     KURTOSIS    1.113204E 01    .50 QUANTILE  (MEDIAN)        1.210000E 03
MIDRANGE    4.306750E 04     RANGE       8.585100E 04     BETA1       8.080197E 12    .75 QUANTILE  (HINGE)         1.015400E 04
GEOM MEAN   1.514564E 03     MIDSPREAD   9.906000E 03     BETA2       5.105574E 17    .90 QUANTILE                  1.864900E 04
HARM MEAN   5.009507E 02                                                              MAXIMUM                       8.599300E 04
```

Figure 3 - TELEPHONE DATA 1. 3rd SECTION.



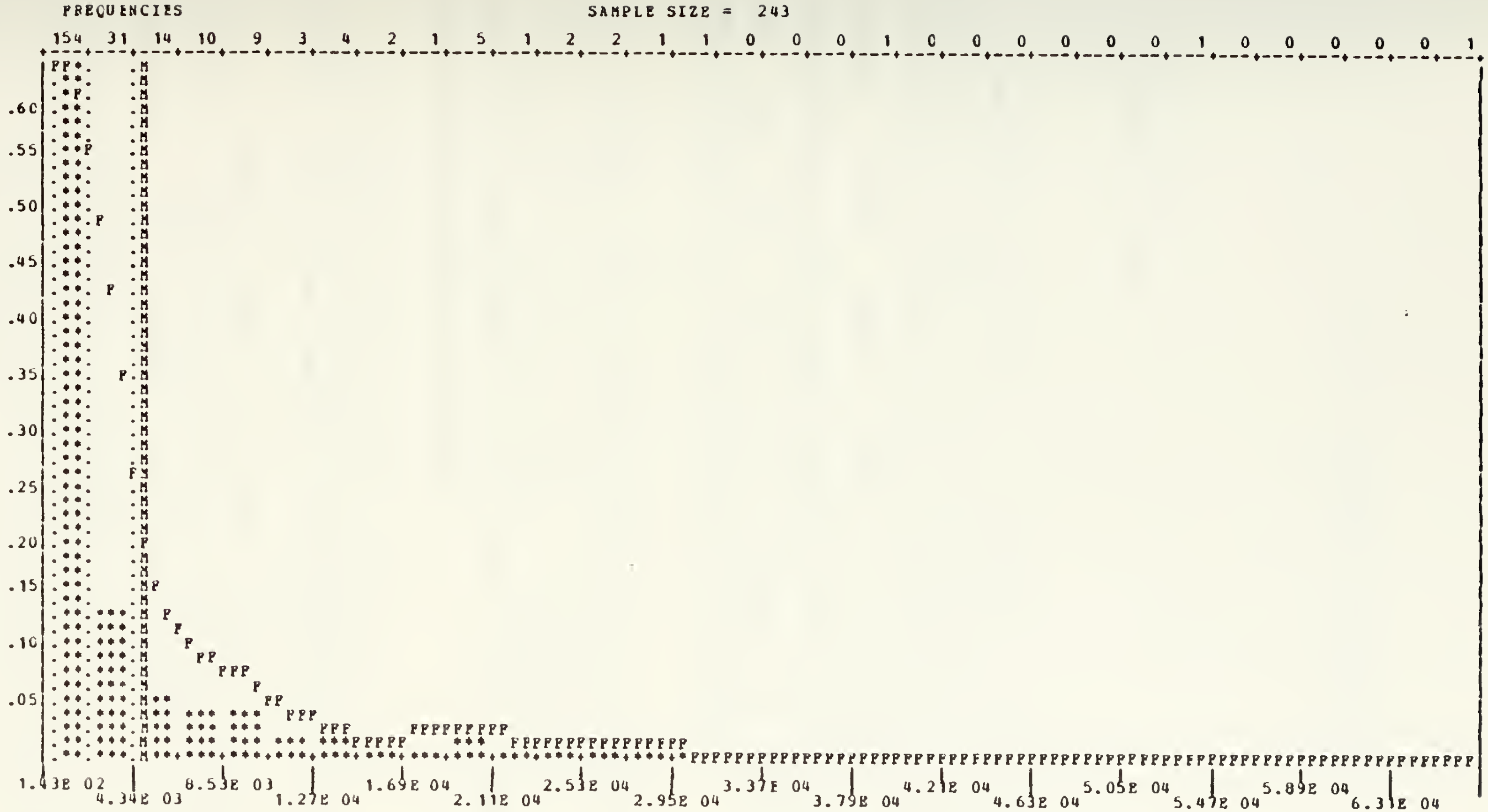
```
CENTRAL TENDENCY              SPREAD                         HIGHER CENTRAL MOMENTS        DISTRIBUTION

MEAN        1.434065E 03      VARIANCE    2.511414E 07       M3          8.845457E 11      MINIMUM                      1.000000E 00
MEDIAN      7.550000E 01      STD DEV     5.011402E 03       M4          4.343909E 16      .10 QUANTILE                 1.000000E 00
TRIMEAN     1.312500E 02      COEF VAR    3.494542E 00       SKEWNESS    7.028177E 00      .25 QUANTILE  (HINGE)        2.000000E 00
MIDMEAN     9.302444E 01      MEAN DEV    1.418856E 03       KURTOSIS    6.587222E 01      .50 QUANTILE  (MEDIAN)       7.550000E 01
MIDRANGE    3.363600E 04      RANGE       6.727000E 04       BETA1       8.809438E 11      .75 QUANTILE  (HINGE)        3.720000E 02
GEOM MEAN   4.910712E 01      MIDSPREAD   3.700000E 02       BETA2       4.320812E 16      .90 QUANTILE                 2.891000E 03
HARM MEAN   3.503419E 00                                                                   MAXIMUM                      6.727100E 04
```

Figure 4 - TELEPHONE DATA 2. THE ENTIRE SET OF DATA.



```
CENTRAL TENDENCY            SPREAD                         HIGHER CENTRAL MOMENTS       DISTRIBUTION

MEAN        6.410158E 01    VARIANCE    2.397849E 03       M3          3.892395E 03     MINIMUM                          2.000000E 00
MEDIAN      6.200000E 01    STD DEV     4.896783E 01       M4          8.039525E 06     .10 QUANTILE                     2.000000E 00
TRIMEAN     6.325000E 01    COEF VAR    7.639099E-01       SKEWNESS    3.315005E-02     .25 QUANTILE (HINGE)             1.000000E 01
MIDMEAN     6.393082E 01    MEAN DEV    4.372697E 01       KURTOSIS   -1.601745E 00     .50 QUANTILE (MEDIAN)            6.200000E 01
MIDRANGE    7.100000E 01    RANGE       1.380000E 02       BETA1       3.855404E 03     .75 QUANTILE (HINGE)             1.190000E 02
GEOM MEAN   3.273445E 01    MIDSPREAD   1.090000E 02       BETA2       8.046432E 06     .90 QUANTILE                     1.250000E 02
HARM MEAN   9.938929E 00                                                                MAXIMUM                          1.400000E 02
```

Figure 5 - TELEPHONE DATA 2. 2nd SECTION.



```
CENTRAL TENDENCY            SPREAD                          HIGHER CENTRAL MOMENTS        DISTRIBUTION

MEAN        4.259676E 03    VARIANCE    6.430278E 07        M3          2.134588E 12      MINIMUM                      1.430000E 02
MEDIAN      1.497000E 03    STD DEV     8.018902E 03        M4          1.055366E 17      .10 QUANTILE                 2.290000E 02
TRIMEAN     1.815500E 03    COEF VAR    1.882514E 00        SKEWNESS    4.139705E 00      .25 QUANTILE  (HINGE)        3.800000E 02
MIDMEAN     1.588683E 03    MEAN DEV    3.721749E 03        KURTOSIS    2.252368E 01      .50 QUANTILE  (MEDIAN)       1.497000E 03
MIDRANGE    3.370700E 04    RANGE       6.712800E 04        BETA1       2.108309E 12      .75 QUANTILE  (HINGE)        3.886000E 03
GEOM MEAN   1.439641E 03    MIDSPREAD   3.508000E 03        BETA2       1.039009E 17      .90 QUANTILE                 1.037600E 04
HARM MEAN   6.280825E 02                                                                  MAXIMUM                      6.727100E 04
```

Figure 6 - TELEPHONE DATA 2. 3rd SECTION.

III. PLOTTING SUBROUTINES

A. INTRODUCTION

Graphical methods of assessing the validity of a probability model, and of estimating some basic parameters (especially location and scale parameters) have been widely applied over many years. A variety of different prescriptions have been advanced for the plotting positions, that is, for the set of values at which the ordered observations in a sample should be plotted.

Plotting the ordered data is an informal and quick visual method for getting an impression of how the distribution of data looks. Depending on what is known about the distribution of the data there are many probability plotting methods.

The Plotting Subroutines presented here, 'NORMPL' and 'EXPLT', use the method of plotting the order statistics ($X_{(i)}$) versus their expected value ($E[X_{(i)}]$) called 'SCORES'. This method does not require knowledge of the Cumulative Distribution Function $F(x)$ (except for its assumed continuity). Therefore it may be very attractive for those who know nothing about the data to be analyzed. In addition the indication of the graph may lead the analyst to further useful results.

Using the Plotting Subroutines 'NORMPL' and 'EXPLT' one may analyze the resulting graphs obtained to get an idea of the distribution of the data.

B. DESCRIPTION OF THE PLOTTING METHOD

The idea behind this method, as above stated, is to plot $X_{(i)}$ versus $E[X_{(i)}]$.

It is used where data $X_1, X_2, \ldots, X_n$ arise as independent observations of a continuous Random Variable X with a distribution function which is believed to have some particular form F(x). Then the ordered data $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ are plotted against $E[X_{(i)}]$.

If the model for F(x) used to get $E[X_{(i)}]$ is true then the resulting plot is a linear (regression) relationship and should plot, within limits of sample variations, as a straight line. Barnett [2] gives the value of $E[X_{(i)}]$ as a linear relation of the plotting position $x_{(i)}$. That is:

$$E[X_{(i)}] = m + sx_{(i)} \quad ,$$

here m, and s, are the location and scale parameters. The Mean and Variance of X are not necessary. As we see the above relation uniquely defines the plotting position as :

$$x_{(i)} = \frac{E[X_{(i)} - m]}{s}$$

$$= E[\frac{X_{(i)} - m}{s}] \quad .$$

Therefore, if the model is correct the linearity will probably be evident. In addition if the resulting plot not being linear leads us to reject the model, we may get an indication of the type of departure present and thus an indication of what action to take or what alternative model to use. It should be noted that the linearity of the graph suggests informally that the model is true, but not to accept the model. Additional formal tests must be done in order to accept the model.

C. SUBROUTINE NORMPL

1. Description

Subroutine 'NORMPL', is used to test a sample of independent observations of a continuous Random Variable X having a distribution function which is believed to have the Normal form.

'NORMPL' sorts the data into increasing order to create a set of Order Statistics. Then the Order Statistics are plotted versus either Normal Scores (if $n \leq 50$, n = number of data-points) or inverse of the Standard Normal Cumulative Distribution Function $(F^{-1}(y))$, (if $n > 50$).

Evaluation of Normal Scores is based on computing the Expected Value of the i^{th} Order Statistic ($E[X_{(i)}]$) .

This evaluation follows:

Let

$g(x_{(i)})$ be the density function of the i^{th} Order Statistic of a sample $X_1, X_2, \ldots, X_n$ from a population having the Standard Normal Distribution F. Then $g(x_{(i)})$ is given by :

$$g(x_{(i)}) = \frac{n!}{(i-1)!\ (n-i)!} F_X(x_{(i)})^{i-1} (1-F_X(x_{(i)}))^{n-i} f_X(x_{(i)})$$

$$= \frac{n!}{(i-1)!\ (n-i)!} P^{i-1} Q^{n-i} Z \ ,$$

where:

$$Z = f_X(x_{(i)}) = \frac{1}{(2\pi)^{1/2}} e^{-x_{(i)}^2/2} \ ,$$

$$P = F_X(x_{(i)}) = \int_{-\infty}^{x_{(i)}} \frac{1}{(2\pi)^{1/2}} e^{-u^2/2}\, du \ ,$$

$$Q = 1-P \ .$$

Now

$$E[X_{(i)}] = \int_{-\infty}^{\infty} x\, g(x)\, dx$$

$$(1) \qquad = \frac{n!}{(i-1)!\ (n-i)!} \int_{-\infty}^{\infty} x\, Z\, P^{i-1} Q^{n-i}\, dx \ .$$

The value of expression (1) has been evaluated and is given in [19]. This table gives the values of Normal Scores from 1 to n/2 (for $n \leq 50$). The values from $(n/2)+1$ to n are evaluated using the symmetric expression :

$$E[X_{(i)}] = - E[X_{(n-i+1)}] \quad .$$

Also

$$E[X_{(k+1)}] = 0, \qquad n = 2k+1 \quad ,$$

'NORMPL' uses the above Normal scores if $n \leq 50$. For $n > 50$ 'NORMPL' computes its own Normal Scores using the program function 'INVNRM', which gives the value x of the inverse Standard Normal distribution ($x = F^{-1}(y)$, $0 \leq y \leq 1$). Justification of this is the fact that if the data are really normally distributed then $E[X_{(i)}]$ $F(i/(n+1))$. Therefore 'NORMPL' can accomodate any sample size.

In addition to the plot given by 'NORMPL', the Wilk-Shapiro test for normality can be evaluated by the same Subroutine. A detailed description of that test is given in [22]. However a summary of that test is presented here in order to help us to describe how the program works. The Wilk-Shapiro test is based on the statistic:

$$W = \frac{b^2}{S^2} \quad ,$$

where:

$$b = \sum_{i=1}^{k} a_{n-i+1} (X_{(n-i+1)} - X_{(i)}), \qquad k=n/2, \qquad n \text{ even}$$

$$b = a_n(X_{(n)}-X_{(1)}) + a_{n-1}(X_{(n-1)}-X_{(2)}) + \ldots$$

$$+ a_{k+2}(X_{(k+2)}-X_{(k)}), \qquad k=(n-1)/2, \quad n \text{ odd}$$

$$S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2$$

The coefficients (a_i) are defined by:

$$a_i = \sum_{j=1}^{n} m_j u_{ij} / C , \qquad i = 1,2,\ldots,n$$

where:

$m_j = E[X_{(j)}]$, (That is the 'NORMAL SCORES')

$u_{ij} = Cov[X_{(i)},X_{(j)}]$, $i,j = 1,2,\ldots,n$

$$C^2 = m' V^{-1} V^{-1} m$$

where:

$$m' = (m_1,m_2,\ldots m_n)$$

$V = (u_{i,j})$, That is the nXn Covariance Matrix.

There are some approximations associated with the evaluation of these coefficients. However for $n \leq 50$ the (a_i) are tabulated in the above-mentioned paper and the program uses them by FORTRAN DATA initialization statements.

For $n>50$ the (a_i) are computed by the program using the following approximation method:

$$a_i = 2m_i/C, \qquad i = 1,2,\ldots,n-1 \quad ,$$

where C is given by the following Least-Square equation as a function of n:

$$C^2 = -2.722 + 4.083n \quad .$$

Also, for n=1

$$a_1^2 = \frac{\Gamma(n/2 + .5)}{2^{1/2}\,\Gamma(n/2 + 1)} \quad .$$

Note that $a_i = a_{n-i+1}$.

Some properties for the W Statistic can be given here, which have been taken from the above mentioned paper:

(a). W is scale and origin invariant.

(b). W has a distribution which depends only on the sample size n, for samples from a Normal Distribution.

(c). W is statistically independent of S^2 and of $\bar{X}$ for samples from a Normal Distribution.

(d). $E[W^r] = E[b^{2r}]/E[S^{2r}]$ for any r.

(e). The maximum value of W is 1 .

(f). The minimum value of W is $na_1^2/(n-1)$.

(g). The half and first moments of W are given by:

$$E[W^{1/2}] = \frac{R^2\,\Gamma(n/2 -.5)}{C\,\Gamma(n/2)\,2^{1/2}} \quad ,$$

$$E[W] = \frac{R^2(R^2+1)}{C^2(n-1)} \quad .$$

where:

$$R^2 = m' V^{-1} m$$

(h). For n=3 the density of W is given as:

$$f_W(w) = \frac{3}{\pi} (1-w)^{-1/2} w^{-1/2}, \qquad .75 \le w \le 1.$$

2. Program Structure

'NORMPL' is a FORTRAN-callable Subroutine with each call returning (optionally) either a plot for a given set of data or a value for the W Statistic or both.

The program is divided into three parts:

The first part is the control program of 'NORMPL'. In that part the 'Normal Scores' for n = 2(1)25 and n = 26(2)50 are stored along with the Subroutine by DATA statements; on the other hand, the Normal Scores for n > 50 are computed by calling the program's function 'INVNRM'. In this part the user specifies just a plot or just the value of W or both. The Library-Subroutine 'PXSORT' is used by the program.

The second part of the program consists of the Subroutine 'PLOT' which accepts any data set to be tested for Normality. 'PLOT' itself scales the data according to its range and plots the scaled data along 110 equal spaced positions of the X-axis. No plot is given if the data has a constant value.

The third part of the program is the 'WILK' Subroutine. In this Subroutine the coefficients (a_i) (for $n \le 50$) of the W Statistic are stored using FORTRAN

DATA initialization statements. For n>50 the coefficients (a_i) are computed by the Subroutine itself. This Subroutine also calls the 'INVNRM' Function to get the 'Normal Scores' to be used for the computation of the (a_i)'s when $n > 50$.

A complete description of how 'NORMPL' operates is given in the Subroutine. However a summary is obtained by typing on the terminal DESCRIBE NORMPL. When the user types the command DESCRIBE NORMPL under the CMS environment the following response is printed on the terminal:

SUBROUTINE NORMPL

'NORMPL' takes a set of data, sorts it into ascending order and uses the created Order Statistics for:

1. Plotting $X_{(i)}$ versus either 'Normal Scores' (if $n \le 50$, n = Sample size) or Inverse of the unit Normal Cumulative Distribution (that is:

$$X_{(i)} \text{ versus Normal scores or } F^{-1}(\, i/(n+1)\,)\ ,)$$

to see if there is a linear fit.

2. Computing the value of the W test Statistic (Wilk-Shapiro test for normality).

It is called by:

CALL NORMPL (X, SCORES, N, K)

ARGUMENTS

X	Is the array containing the data
SCORES	Is a work array of dimension N
N	Is the number of data values
K	User's option For: K = 1 A plot only ,

K = 2 The W-value is given ,
K = 3 Both of the above are given.

More information is given in the subroutine.

3. Interpreting The Output

If data are really Normally distributed then a straight line is expected from the output. This is an indication that gives the user a first feeling of the distribution of data. But if the plot is not linear then the user has to reject the normality assumption. In addition the shape of the plot may lead the user to make an alternative decision of what model he may use. For example:

(a). Shape of the form of Figure 7a suggests that data are skewed to one side (specifically positive values). In that case an Exponential model could be an appropriate one. The histogram of the data is given in Figure 7b.

(b). Shape of the form of Figure 8a can be interpreted as data not as dispersed as the Normal Distribution. This set of data may have a Uniform or a Triangular Distribution, and could, for instance, arise if measuring tolerances of components hand-picked to lie within tolerances. The associated histogram is given in Figure 8b.

(c). Shape of the form of Figure 9a indicates much more dispersed data than Normal, but the data is symmetric, probably coming from a Double Exponential or Cauchy Distribution. The associated histogram is given in Figure 9b.

It should be noted here that a straight line does not always arise from Normal data. Figure 10a for example

gives an indication that data are Normal (Straight line). But these data have been generated by computer from a Symmetric Triangular Distribution in the range (0 to 2). On the other hand figure 11a plots a data set of size 50 again from a Triangular (0 to 2) distribution. But in this case we observe that the graph is not a straight line. These results demonstrate the informality of that Probability Plotting Method and in these cases the user has to continue testing the data by other formal tests. Nevertheless, since this method is primarily an initial screening device, ease of application is important. Figure 11b gives the associated histogram.

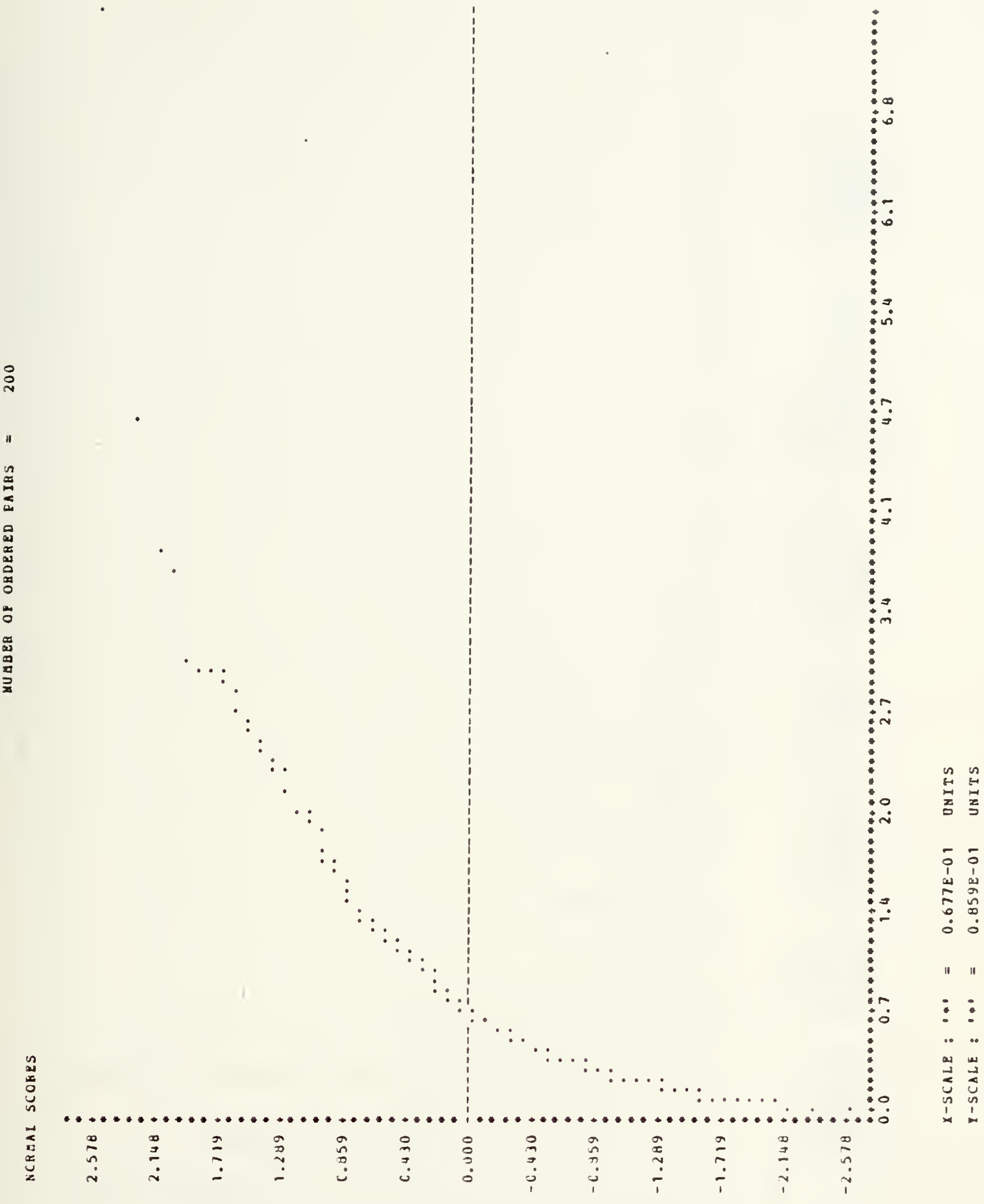
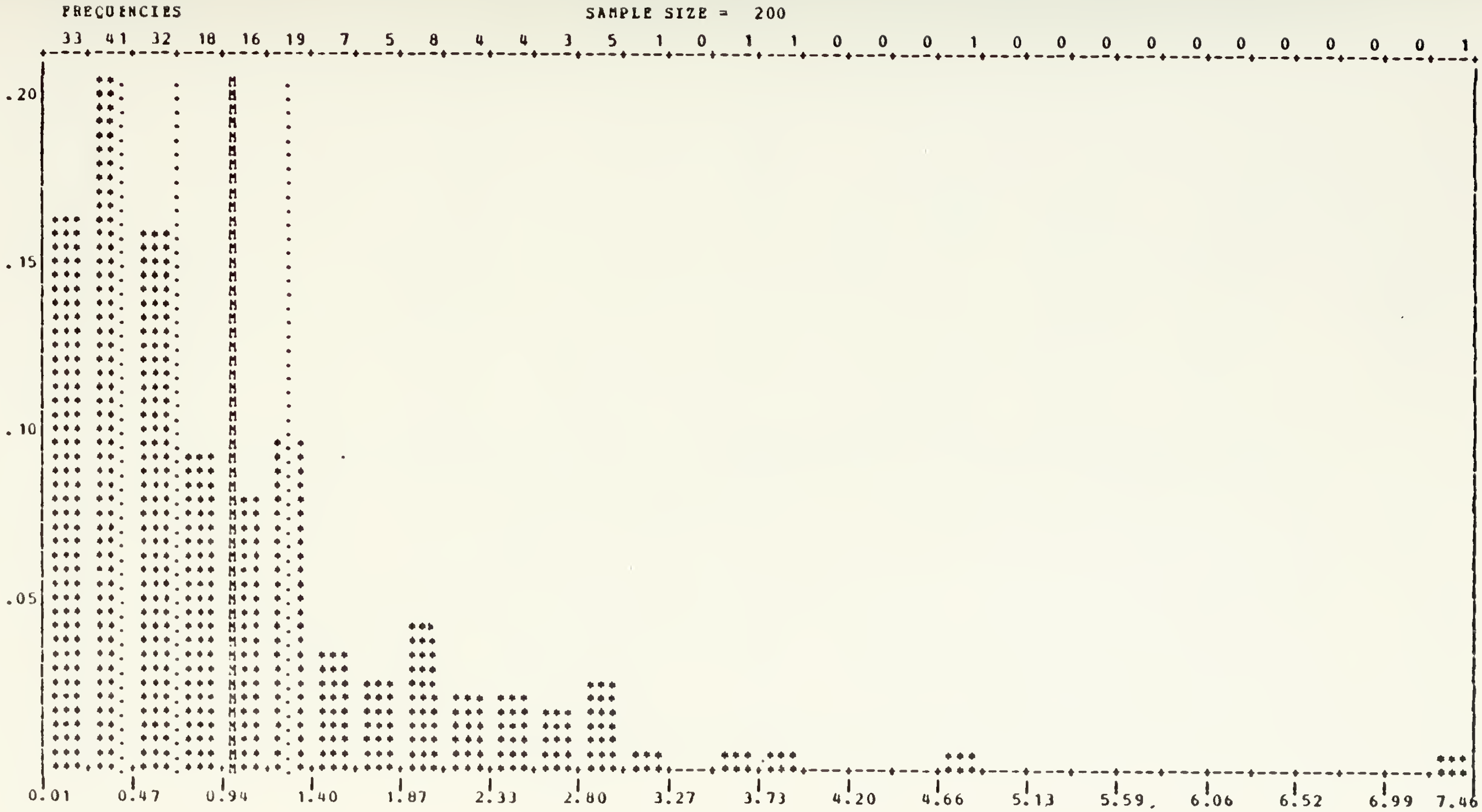


Figure 7a - GENERATED DATA FROM AN EXPONENTIAL DISTRIBUTION



```
CENTRAL TENDENCY                SPREAD                       HIGHER CENTRAL MOMENTS      DISTRIBUTION

MEAN         9.603943E-01       VARIANCE    8.978770E-01     M3         2.215768E 00     MINIMUM                        6.307125E-03
MEDIAN       6.498733E-01       STD DEV     9.475637E-01     M4         1.161485E 01     .10 QUANTILE                   1.647375E-01
TRIMEAN      7.208295E-01       COEF VAR    9.866402E-01     SKEWNESS   2.604347E 00     .25 QUANTILE  (HINGE)          3.331240E-01
MIDMEAN      7.158095E-01       MEAN DEV    6.249905E-01     KURTOSIS   1.140721E 01     .50 QUANTILE  (MEDIAN)         6.498733E-01
MIDRANGE     3.731236E 00       RANGE       7.449860E 00     BETA1      2.182643E 00     .75 QUANTILE  (HINGE)          1.250448E 00
GEOM MEAN    6.038144E-01       MIDSPREAD   9.173242E-01     BETA2      1.140656E 01     .90 QUANTILE                   2.155207E 00
HARM MEAN    2.471905E-01                                                                MAXIMUM                        7.456167E 00
```

Figure 7b - HISTOGRAM OF THE DATA OF FIGURE 7a

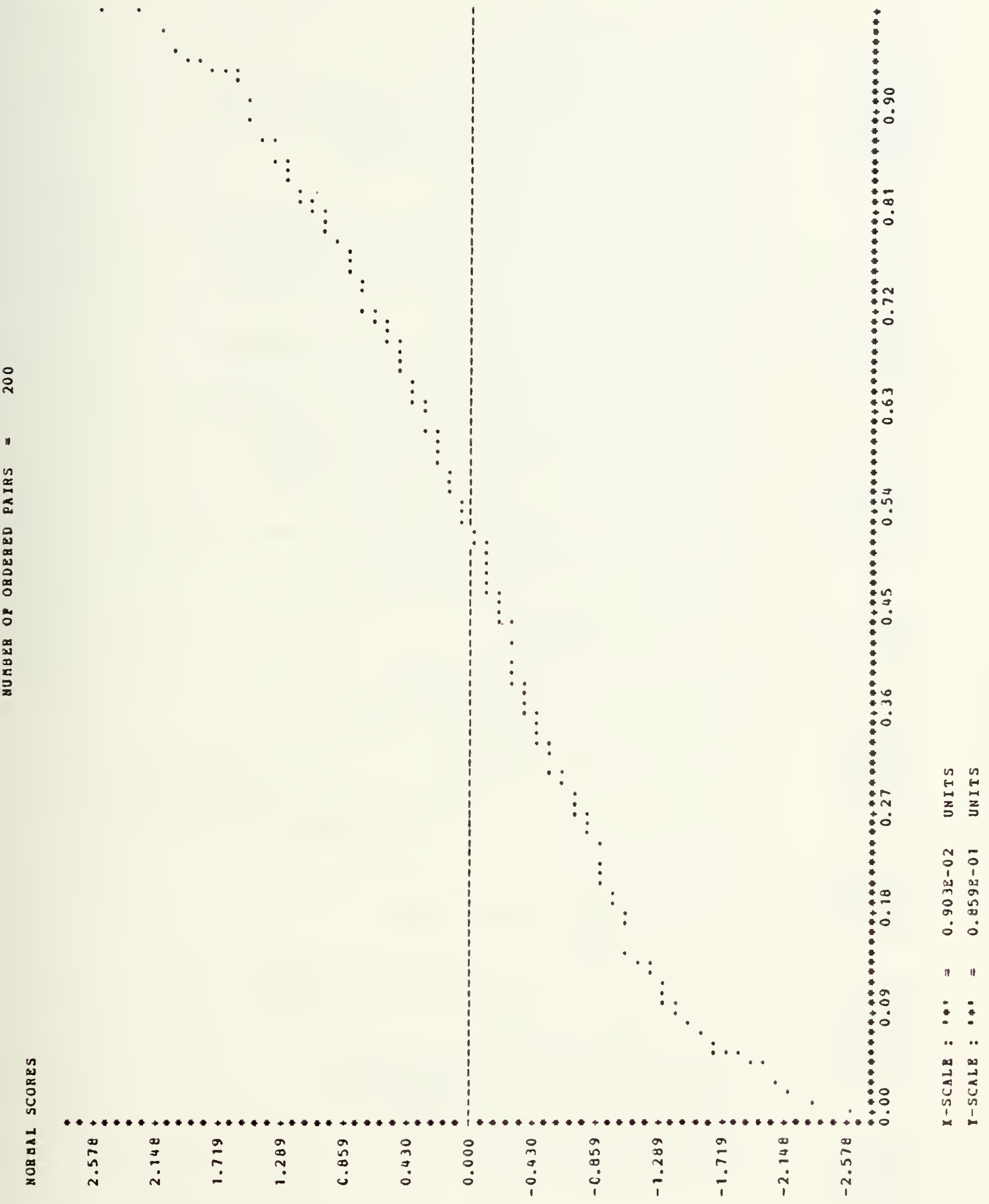
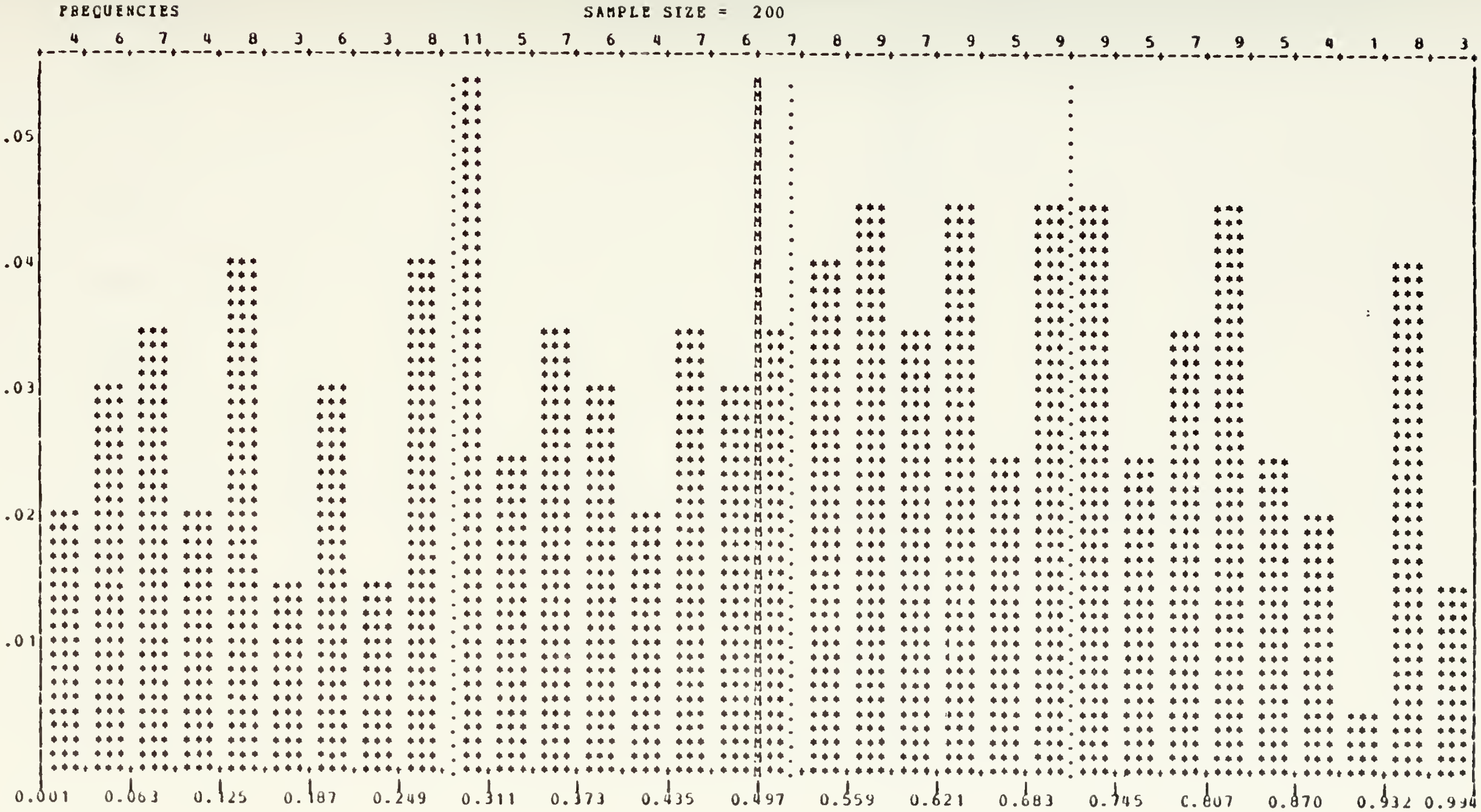


Figure 8a - GENERATED DATA FROM A UNIFORM (0,1) DISTRIBUTION



```
CENTRAL TENDENCY               SPREAD                       HIGHER CENTRAL MOMENTS      DISTRIBUTION

MEAN        4.982966E-01       VARIANCE   7.142746E-02      M3         -1.559286E-03    MINIMUM                      5.778670E-04
MEDIAN      5.221119E-01       STD DEV    2.672592E-01      M4          9.784468E-03    .10 QUANTILE                 1.121920E-01
TRIMEAN     5.118222E-01       COEF VAR   5.363456E-01      SKEWNESS   -8.168226E-02    .25 QUANTILE    (HINGE)      2.863849E-01
MIDMEAN     5.065845E-01       MEAN DEV   2.280766E-01      KURTOSIS   -1.082185E 00    .50 QUANTILE    (MEDIAN)     5.221119E-01
MIDRANGE    4.971453E-01       RANGE      9.931349E-01      BETA1      -1.535976E-03    .75 QUANTILE    (HINGE)      7.166810E-01
GEOM MEAN   3.827419E-01       MIDSPREAD  4.302961E-01      BETA2       9.740688E-03    .90 QUANTILE                 8.388529E-01
HARM MEAN   7.714224E-02                                                                MAXIMUM                      9.937127E-01
```

Figure 8b - HISTOGRAM OF THE DATA OF FIGURE 8a

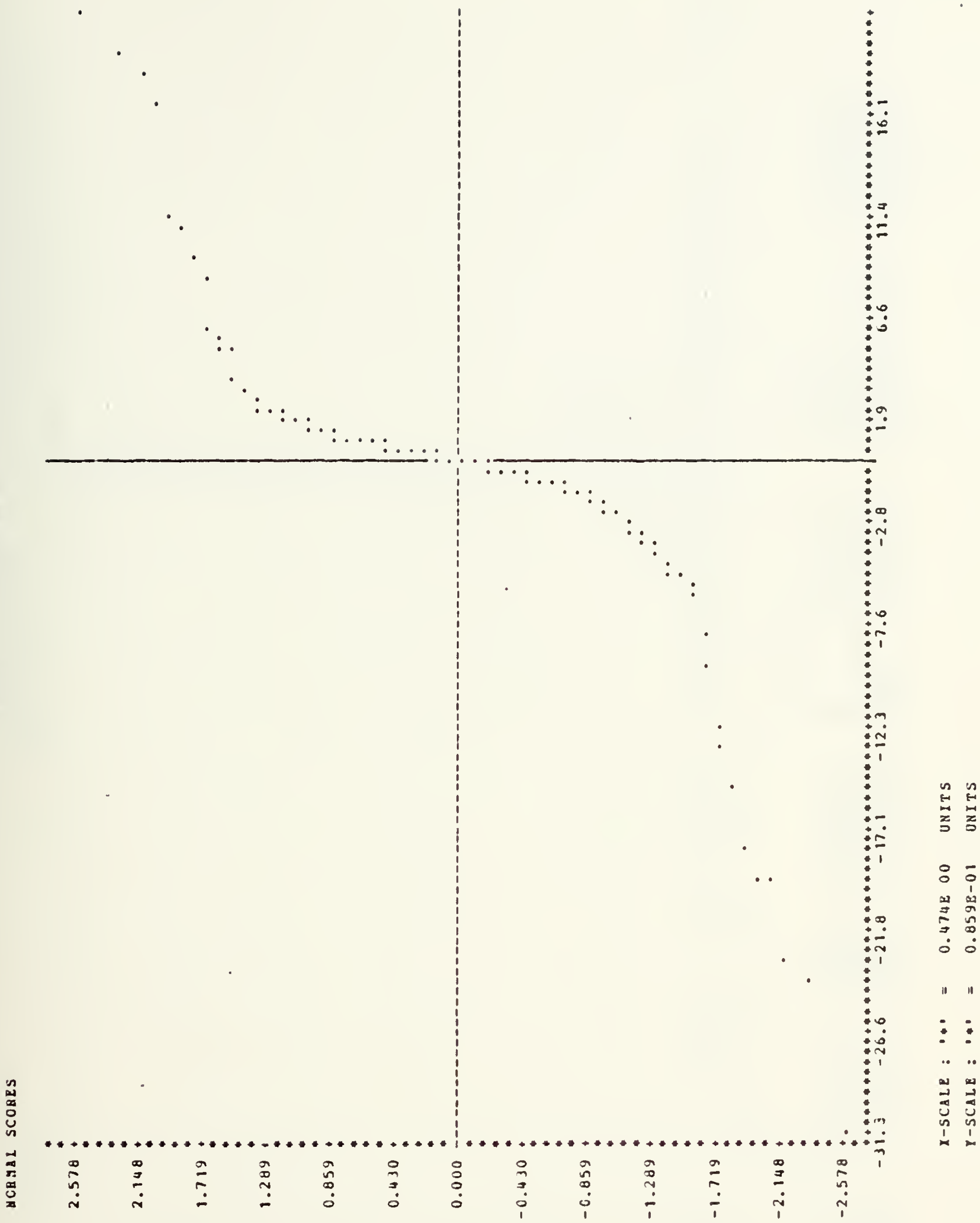
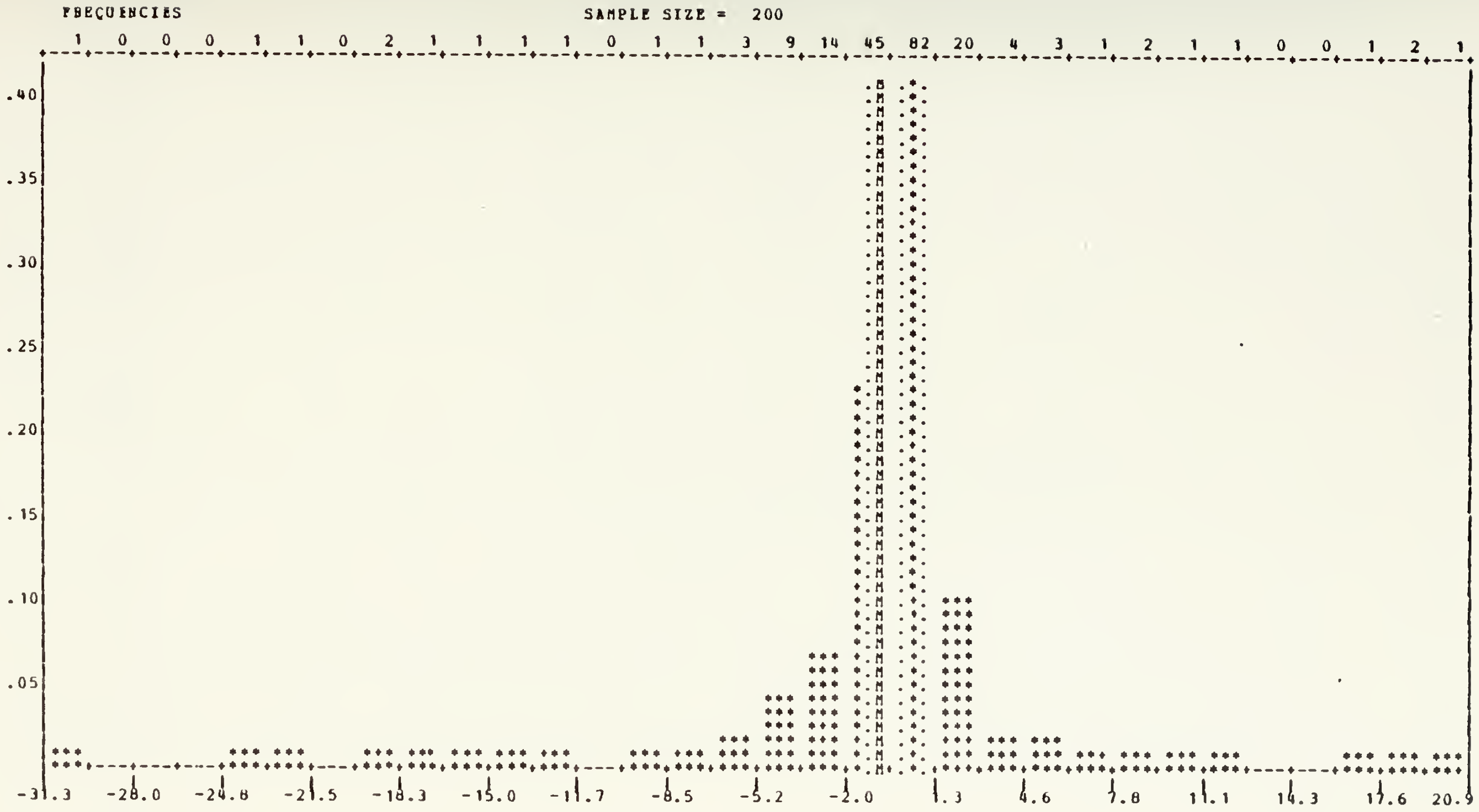


Figure 9a - GENERATED DATA FROM A CAUCEY DISTRIBUTION



```
CENTRAL TENDENCY             SPREAD                        HIGHER CENTRAL MOMENTS      DISTRIBUTION

MEAN       -5.279933E-01     VARIANCE    3.192226E 01      M3         -2.401164E 02    MINIMUM                       -3.130643E 01
MEDIAN     -3.472334E-02     STD DEV     5.649979E 00      M4          1.297994E 04    .10 QUANTILE                  -4.170311E 00
TRIMEAN    -7.294822E-02     COEF VAR    1.070085E 01      SKEWNESS - 1.331316E 00     .25 QUANTILE (HINGE)          -1.190001E 00
MIDMEAN    -4.751503E-02     MEAN DEV    2.799565E 00      KURTOSIS    9.737538E 00    .50 QUANTILE (MEDIAN)         -3.472334E-02
MIDRANGE   -5.219429E 00     RANGE       5.217400E 01      BETA1      -2.365266E 02    .75 QUANTILE (HINGE)           9.676542E-01
                             MIDSPREAD   2.157655E 00      BETA2       1.275069E 04    .90 QUANTILE                   2.190511E 00
                                                                                       MAXIMUM                        2.086757E 01
```

Figure 9b - HISTOGRAM OF THE DATA OF FIGURE 9a

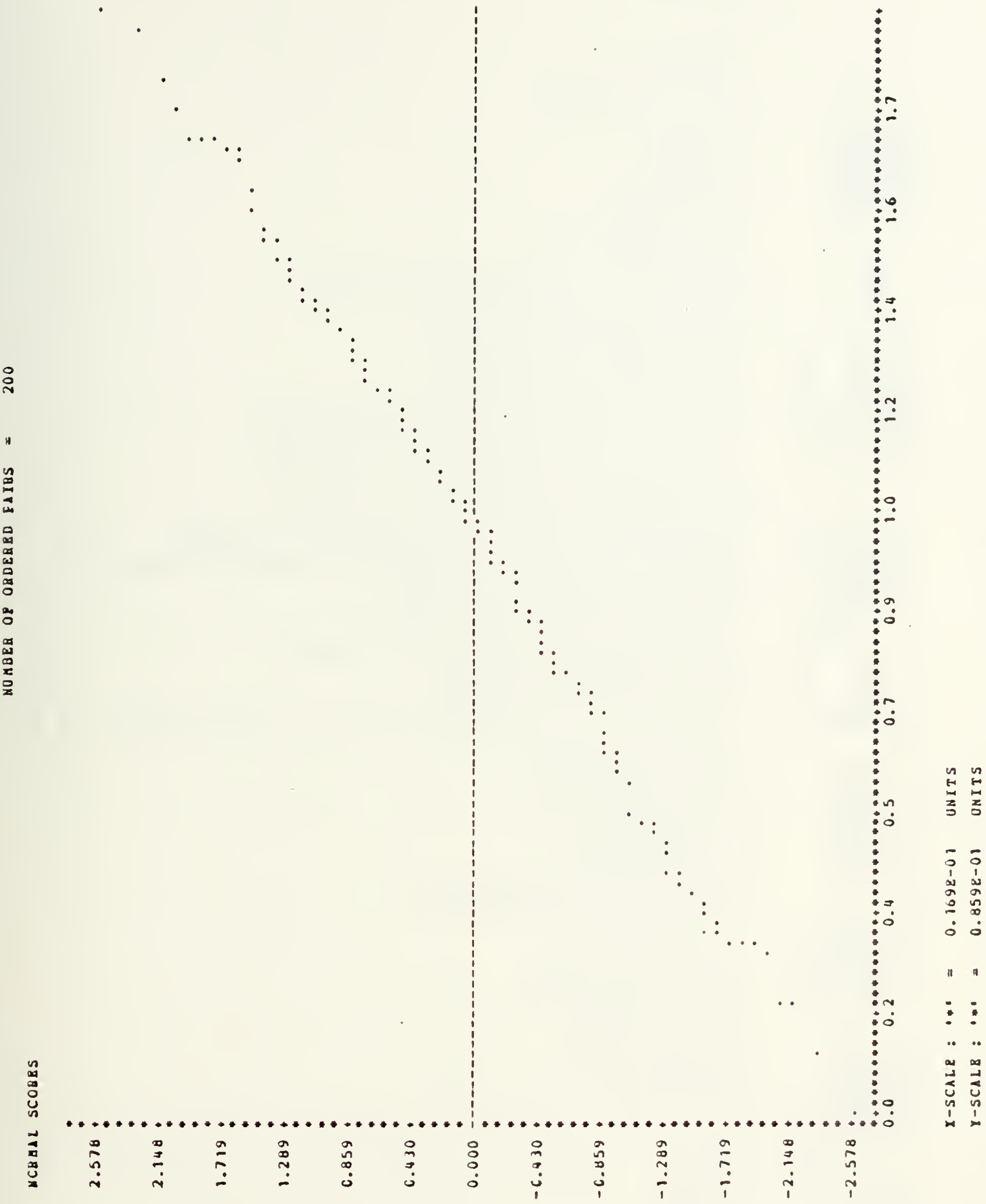
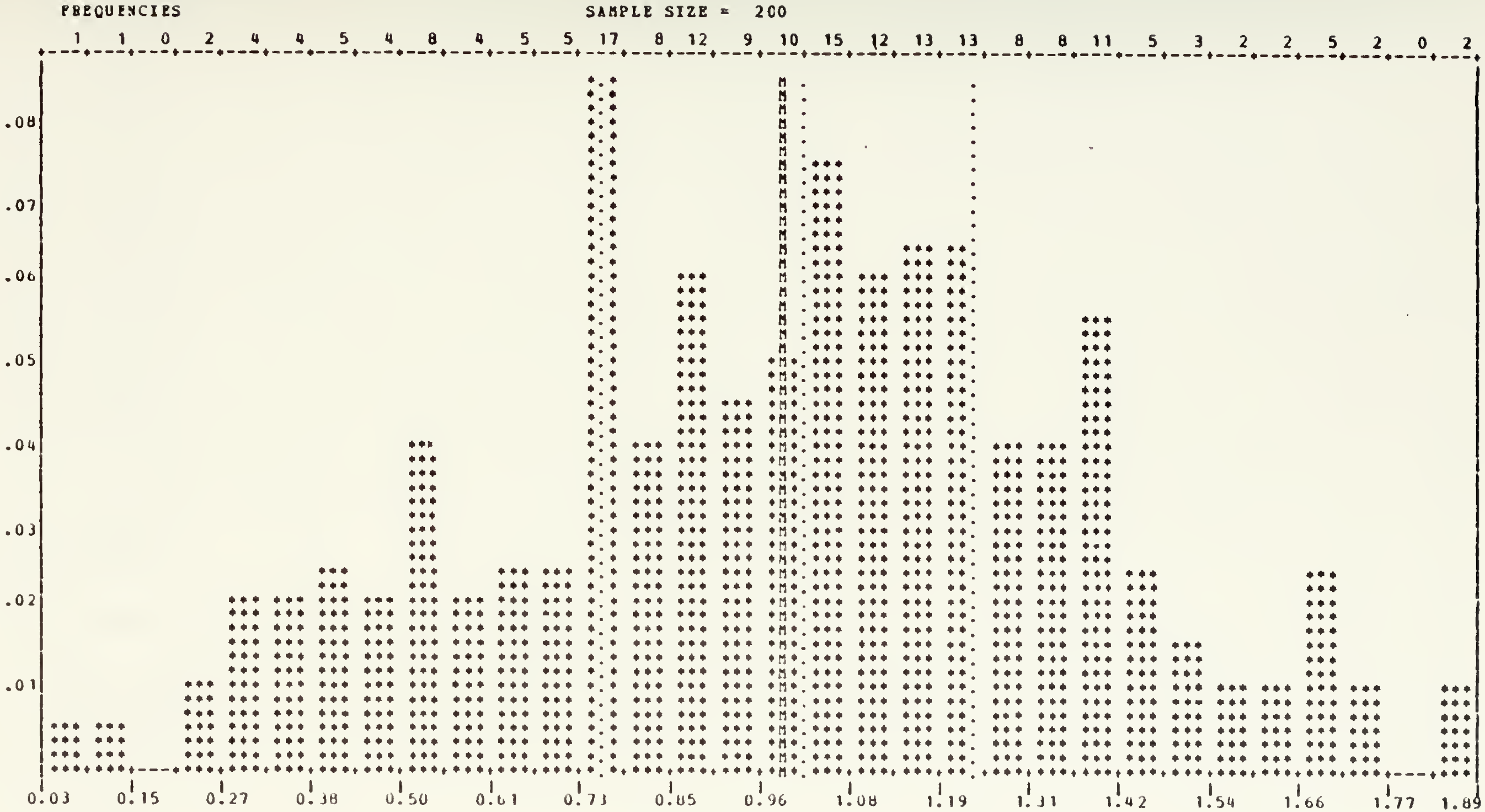


Figure 10a - GENERATED DATA FROM A TRIANGULAR (0,1) DISTRIBUTION. SAMPLE SIZE 200



```
CENTRAL TENDENCY             SPREAD                       HIGHER CENTRAL MOMENTS      DISTRIBUTION

MEAN        9.933041E-01     VARIANCE    1.351810E-01     M3         -5.775355E-03    MINIMUM                       3.399608E-02
MEDIAN      1.022362E 00     STD DEV     3.676697E-01     M4          4.820189E-02    .10 QUANTILE                  4.736916E-01
TRIMEAN     1.012195E 00     COEF VAR    3.701482E-01     SKEWNESS   -1.161998E-01    .25 QUANTILE  (HINGE)         7.568092E-01
MIDMEAN     1.006981E 00     MEAN DEV    2.953447E-01     KURTOSIS   -3.622561E-01    .50 QUANTILE  (MEDIAN)        1.022362E 00
MIDRANGE    9.609294E-01     RANGE       1.853867E 00     BETA1      -5.689014E-03    .75 QUANTILE  (HINGE)         1.247247E 00
GEOM MEAN   9.029446E-01     MIDSPREAD   4.904375E-01     BETA2       4.778197E-02    .90 QUANTILE                  1.432290E 00
HARM MEAN   7.300129E-01                                                              MAXIMUM                       1.887863E 00
```

Figure 10b - HISTOGRAM OF THE DATA OF FIGURE 10a

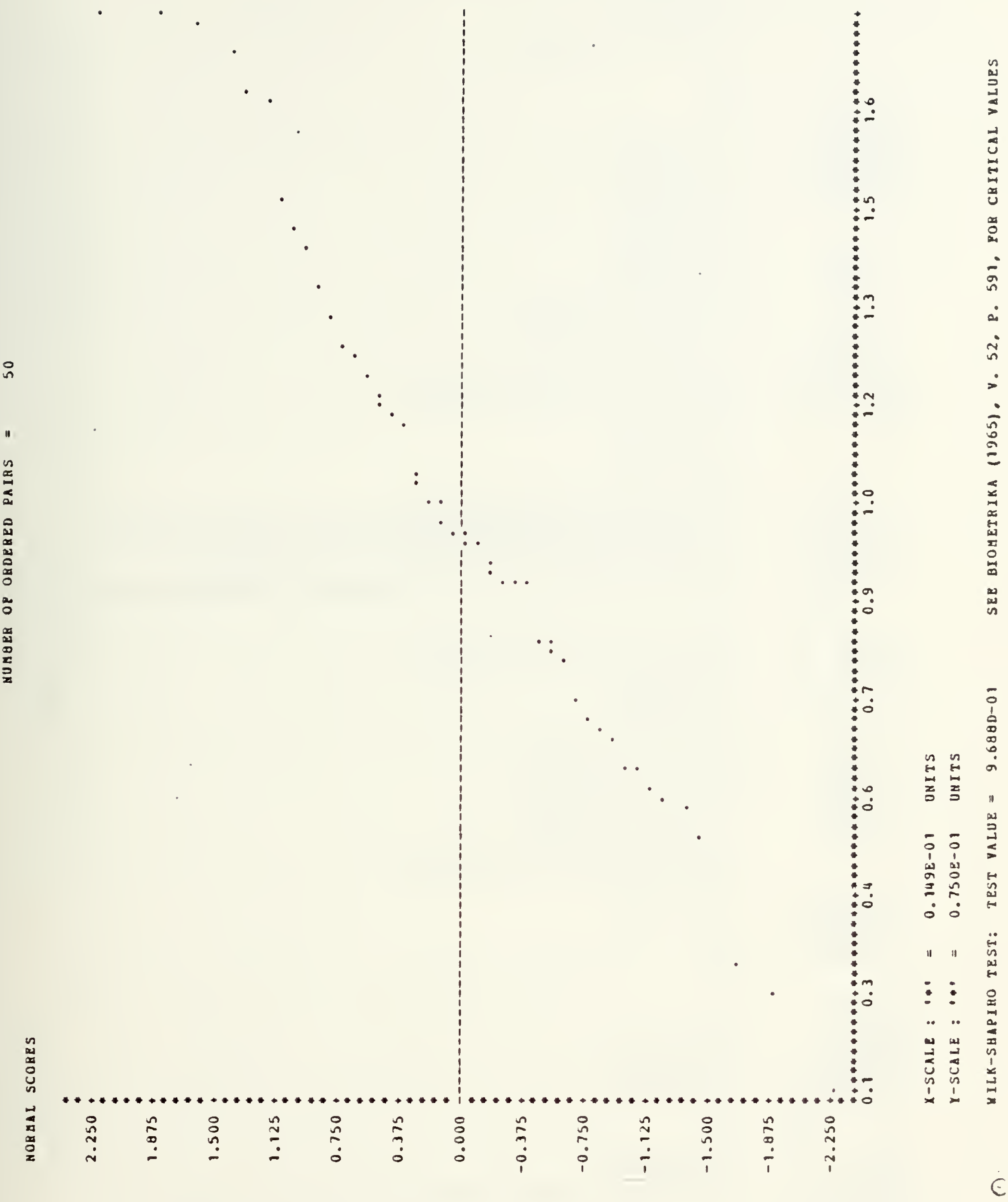
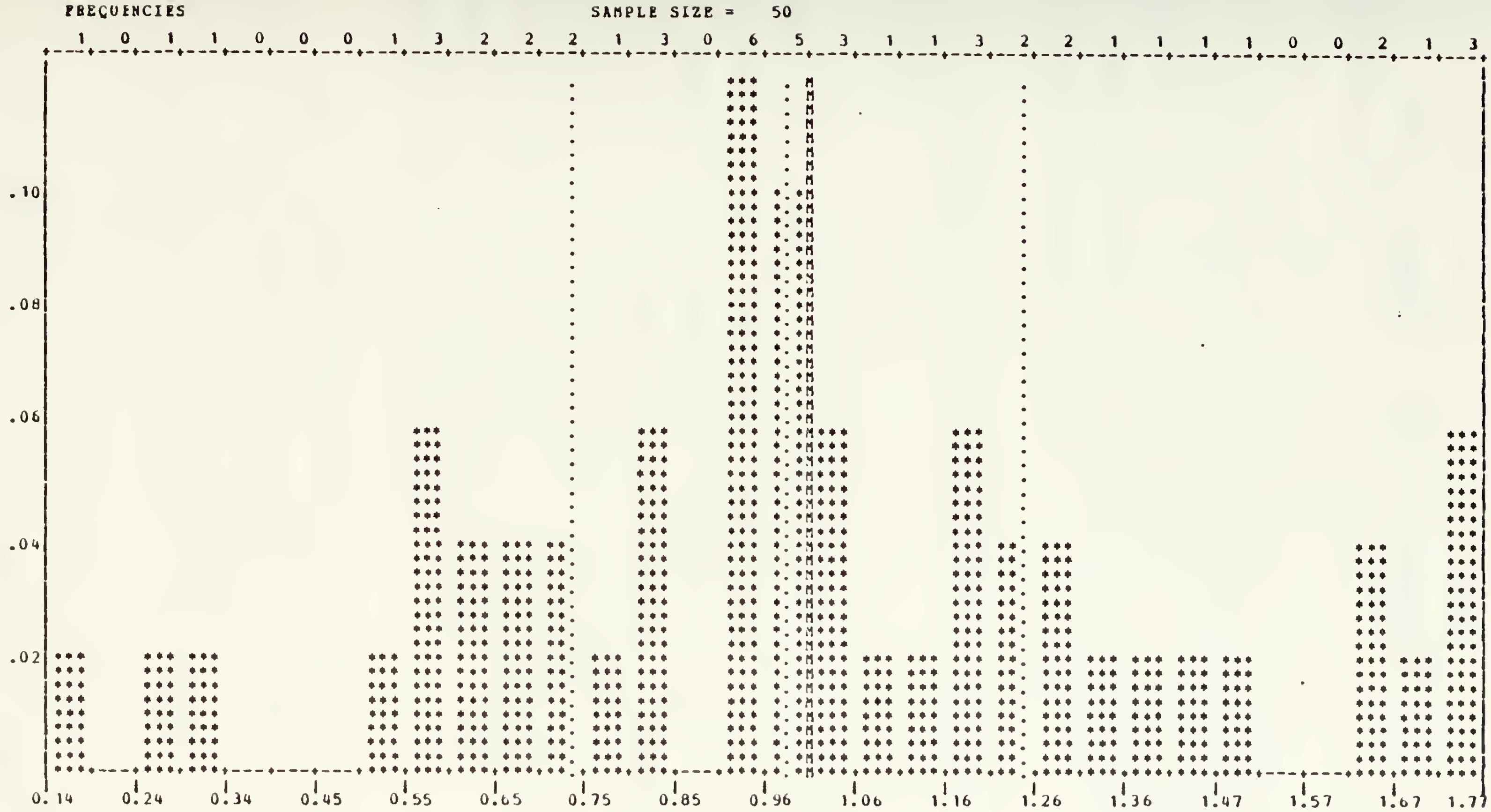


Figure 11a - GENERATED DATA FROM A TRIANGULAR (C,1) DISTRIBUTION. SAMPLE SIZE 50



```
CENTRAL TENDENCY              SPREAD                          HIGHER CENTRAL MOMENTS        DISTRIBUTION

MEAN       1.013797E 00       VARIANCE    1.538159E-01        M3          1.065239E-02      MINIMUM                        1.389797E-01
MEDIAN     9.802237E-01       STD DEV     3.921937E-01        M4          6.290221E-02      .10 QUANTILE                   5.632662E-01
TRIMEAN    9.861524E-01       COEF VAR    3.868563E-01        SKEWNESS    1.765814E-01      .25 QUANTILE  (HINGE)          7.337485E-01
MIDMEAN    9.899505E-01       MEAN DEV    3.025578E-01        KURTOSIS   -3.413353E-01      .50 QUANTILE  (MEDIAN)         9.802237E-01
MIDRANGE   9.560494E-01       RANGE       1.634140E 00        BETA1       1.002176E-02      .75 QUANTILE  (HINGE)          1.250414E 00
GEOM MEAN  9.231459E-01       MIDSPREAD   5.166654E-01        BETA2       6.062489E-02      .90 QUANTILE                   1.642227E 00
HARM MEAN  7.937615E-01                                                                     MAXIMUM                        1.773120E 00
```

Figure 11b - HISTOGRAM OF THE DATA OF FIGURE 11a

4. Using 'NORMPL' with Generated Data from Various Distributions.

The Subroutine 'NORMPL' has been used here to test for Normality some data sets which were generated by the computer from various distributions. The plots of these data sets illustrate the advantages and disadvantages of the plotting Subroutine 'NORMPL'.

Observing Figures 7a-b, 8a-b, 9a-b and 12a-b we may note the following:

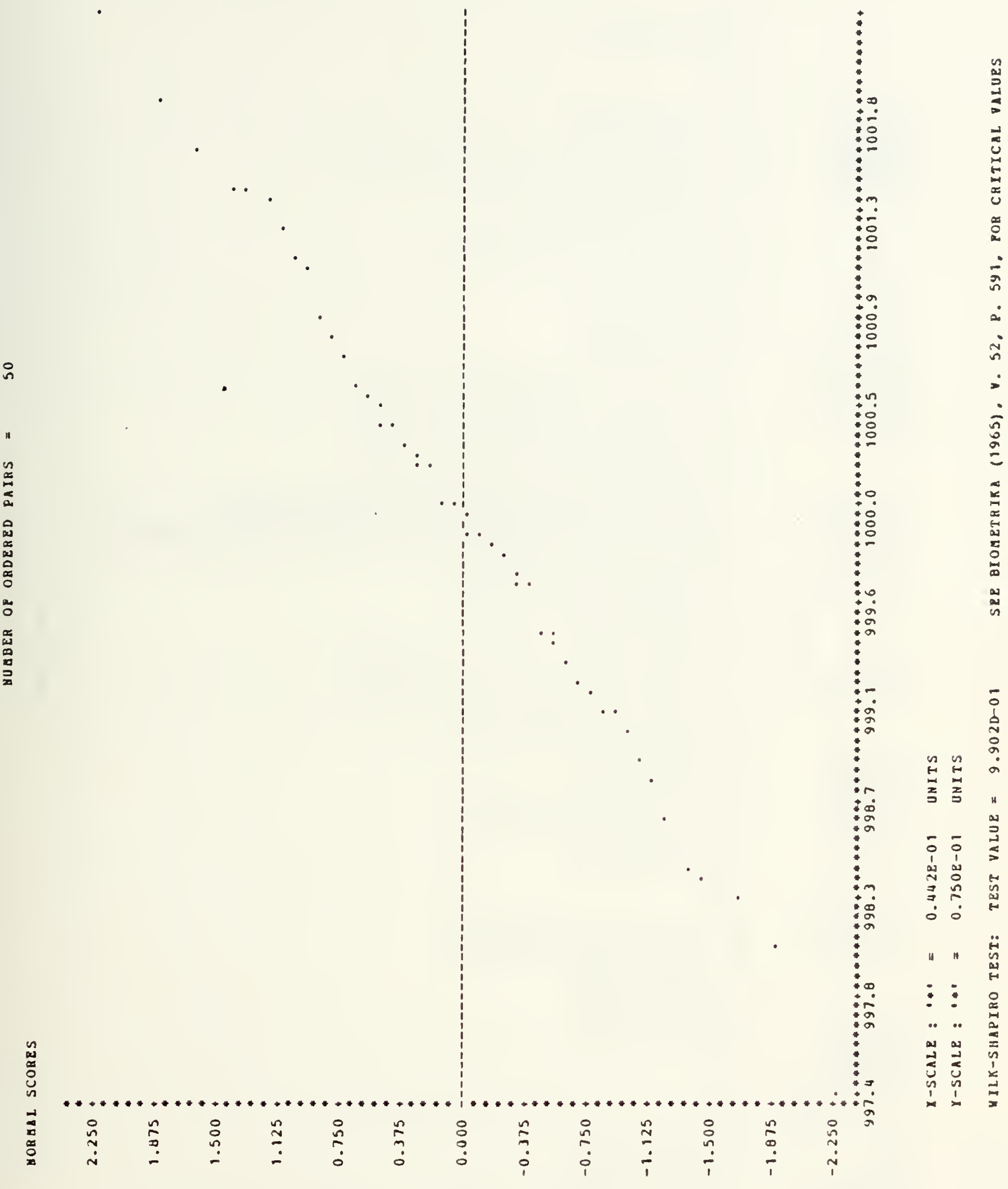
(a). Figure 12a shows a plot of a sample of 50 data points from a Normal Distribution (N(1000,1)). As it was expected a very straight line is fitted. In addition to that plot, the Subroutine 'NORMPL' uses the option to evaluate the 'WILK-SHAPIRO' test and the W-value is computed and printed on the same figure. Comparing this value with the percentage points of the W-test which are given in [22], for sample size n=50 we see that we may accept the normality assumption with a significanse level = .02 . Figure 12b gives the associated histogram.

(b). Figure 7a plots a sample of 200 data points of the Unit Exponential distribution. Here the nonlinear graph is obvious (as is expected) and suggests to reject the normality assumption of the data without doing any further formal test.

(c). Figure 8a gives a plot of a sample of size 200 from Uniform (0, 1) generated data. The shape of that plot indicates data not as dispersed as the normal, therefore the user is told to reject normality, getting at the same time a suggestion that data may be Uniformly

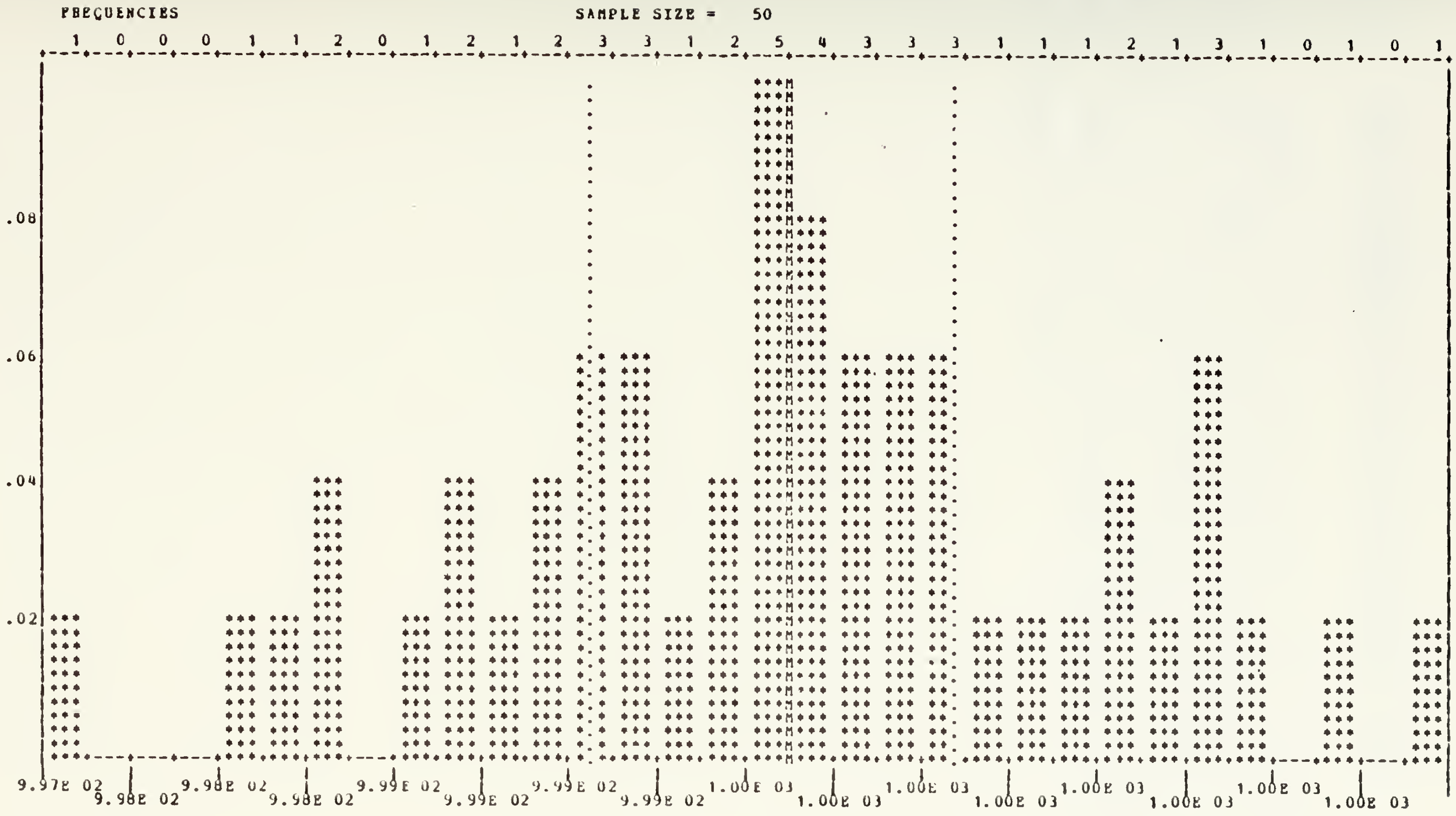
distributed (because of that particular shape of the graph).

(d). Figure 9a suggests strongly departure from a Normality assumption. Indeed this set of data have been generated from a Cauchy Distribution and its shape really indicates a symmetric distribution.



```
X-SCALE : '*' =   0.442E-01   UNITS
Y-SCALE : '*' =   0.750E-01   UNITS

WILK-SHAPIRO TEST:  TEST VALUE =  9.902D-01        SEE BIOMETRIKA (1965), V. 52, P. 591, FOR CRITICAL VALUES
```

Figure 12a - GENERATED DATA FROM A NORMAL (1000,1) DISTRIBUTION



```
CENTRAL TENDENCY              SPREAD                            HIGHER CENTRAL MOMENTS        DISTRIBUTION

MEAN        9.999438E 02      VARIANCE    1.093181E 00          M3          -8.221918E-02     MINIMUM                        9.973699E 02
MEDIAN      9.999368E 02      STD DEV     1.045553E 00          M4           3.268539E 00     .10 QUANTILE                   9.983872E 02
TRIMEAN     9.999148E 02      COEF VAR    1.045612E-03          SKEWNESS    -7.193416E-02     .25 QUANTILE  (HINGE)          9.992476E 02
MIDMEAN     9.999377E 02      MEAN DEV    8.165478E-01          KURTOSIS    -2.649212E-01     .50 QUANTILE  (MEDIAN)         9.999368E 02
MIDRANGE    9.998003E 02      RANGE       4.861084E 00          BETA1       -7.735181E-02     .75 QUANTILE  (HINGE)          1.000538E 03
GEOM MEAN   9.999434E 02      MIDSPREAD   1.290527E 00          BETA2        3.146208E 00     .90 QUANTILE                   1.001402E 03
HARM MEAN   9.999429E 02                                                                      MAXIMUM                        1.002231E 03
```

Figure 12b - HISTOGRAM OF THE DATA OF FIGURE 12a

5. Using 'NORMPL' With Cost Overruns Data

'NORMPL' was used with the cost overruns data [6], for the year 1950 to see if there is a linear fit. Looking at Figure 13 we see that data may be Normally distributed since the plot is very close to linear. Of course some data points deviate from the straight line, but these points may be considered as outliers.

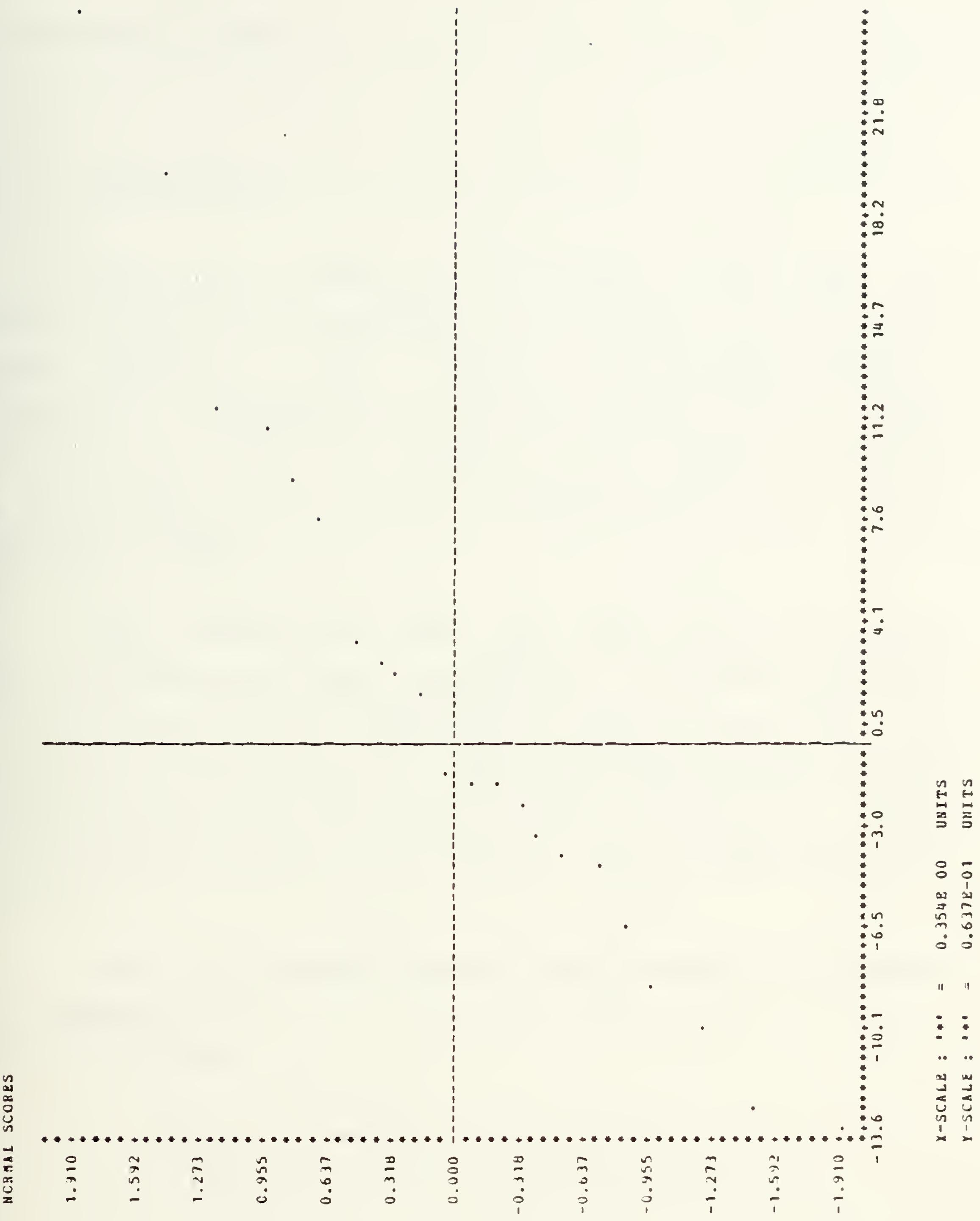


Figure 13 - COST OVERRUNS DATA

D. SUBROUTINE EXPLT

1. Description

Subroutine 'EXPLT' is indended to be used for testing a sample of independent observations of a continuous random variable X with a distribution function which is believed to be Exponential, by plotting each of the Order Statistics $X_{(i)}$ versus its Expected Value, and by estimating the parameters γ_1 and γ_2 of the observations (γ_1, and γ_2 introduced later).

This Subroutine sorts the data into increasing order, obtaining the Order Statistics. The Order Statistics are then plotted versus Exponential Scores. These scores are evaluated by the Subroutine. The Exponential Scores are the expected value of the Order Statistics ($E[X_{(i)}]$). They are more easily derived and computed than the Normal Scores. That is:

Let X be a random variable Exponentially distributed with parameter $\lambda = 1$

Then it is known:

$$f_X(x) = \lambda e^{-\lambda x} , \qquad \lambda, x > 0 ,$$

$$F_X(x) = 1-e^{-\lambda x} , \qquad \lambda, x > 0.$$

The Density of the i^{th} Order Statistic is given by:

$$f_{X_{(i)}}(x_{(i)}) = n\binom{n-1}{i-1} F_X(x_{(i)})^{i-1} (1-F_X(x_{(i)}))^{n-i} f_X(x_{(i)})$$

Thus for the 1^{st} Order Statistic we have:

$$f_{X_{(1)}}(x_{(1)}) = n(1-(1-e^{-\lambda x_{(1)}}))^{n-1} \lambda e^{-\lambda x_{(1)}}$$

$$= n\lambda e^{-n\lambda x_{(1)}} .$$

Therefore $X_{(1)}$ is exponentially distributed with parameter $\lambda' = n\lambda$.

And for $\lambda = 1$ we have:

$$E[X_{(1)}] = \frac{1}{n\lambda} = \frac{1}{n} .$$

It can be proved (see Feller [7]) that the n random variables $X_{(1)}, X_{(2)}-X_{(1)}, X_{(3)}-X_{(2)}, \ldots, X_{(n)}-X_{(n-1)}$ from an Exponential random variable X with density function:

$$f_X(x) = \lambda e^{-\lambda x} , \qquad x > 0 ,$$

are independent and the density of the random variable $Y = X_{(k)} - X_{(k-1)}$ is given by:

$$f_Y(y) = (n-k+1)\lambda e^{-\lambda(n-k+1)y} , \qquad y > 0 .$$

That is, Y is an Exponential random variable with parameter $(r-k+1)\lambda$.

Therefore E[Y] is given by:

$$E[X_{(k)} - X_{(k-1)}] = \frac{1}{\lambda(n-k+1)} = \frac{1}{n-k+1}, \qquad \text{for} \quad =1 \quad .$$

This fact is used to derive $E[X_{(i)}]$, $i = 2,3,\ldots,n$.

That is:

$$E[X_{(i)}) = E[\ (X_{(i)} - X_{(i-1)}) + (X_{(i-1)} - X_{(i-2)}) + \ldots +$$

$$(X_{(2)} - X_{(1)}) + X_{(1)}\ ]$$

$$= E[X_{(1)}] + E[X_{(2)} - X_{(1)}] + \ldots +$$

$$E[X_{(i-1)} - X_{(i-2)}] + E[X_{(i)} - X_{(i-1)}]$$

$$= \frac{1}{n} + \frac{1}{n-2+1} + \frac{1}{n-3+1} + \ldots + \frac{1}{n-i+1}$$

$$= \sum_{k=1}^{i} \frac{1}{n-k+1}, \qquad i = 2,\ldots n \quad .$$

'EXPLT' derives the values of $E[X_{(i)}]$ using the above formula and then it uses its Subroutine 'EPLOT' to plot the Order Statistics versus their expected value.

In addition to the plot, which is an informal and quick test for exponentiality the estimates of the statistics γ_1 and γ_2 provide another informal test since the values of γ_1 and γ_2 have a constant value for any Exponential distribution:

By definition γ_1 and γ_2 are given by:

$$\gamma_1 = E\left[\frac{(X-\mu)^3}{\sigma^3}\right] ,$$

$$\gamma_2 = E\left[\frac{(X-\mu)^4}{\sigma^4}\right] - 3 .$$

Thus

$$\gamma_1 = \frac{1}{\sigma^3} E[(X-\mu)^3]$$

$$= \frac{1}{\sigma^3} E[X^3 - 3X^2\mu + 3X\mu^2 - \mu^3]$$

(1) $$= \frac{1}{\sigma^3} (E[X^3] - 3\mu E[X^2] + 3\mu^2 E[X] - \mu^3) .$$

Now

$$E[X^3] = \int_0^\infty x^3 f_X(x)\, dx$$

$$= \int_0^\infty x^3 \lambda e^{-\lambda x}\, dx$$

$$= \frac{\Gamma(4)}{\lambda^3} \int_0^\infty \frac{x^{4-1}}{\Gamma(4)} \lambda^4 e^{-\lambda x}\, dx .$$

The above integral is that of the Gamma density function with parameter $\alpha = 4$, and since its value equals 1 we have:

$$E[X^3] = \frac{3!}{\lambda^3} = \frac{6}{\lambda^3} .$$

Computing $E[X^2]$ we have:

$$E[X^2] = \int_0^\infty x^2 \lambda e^{-\lambda x} dx$$

$$= \frac{\Gamma(3)}{\lambda^2} \int_0^\infty \frac{x^{3-1}}{\Gamma(3)} \lambda^3 e^{-\lambda x} dx$$

$$= \frac{2}{\lambda^2} \quad .$$

Also $E[X] = \frac{1}{\lambda} = \mu$.

Substituting in (1) we have:

$$\gamma_1 = E[\frac{(X-\mu)^3}{3}]$$

$$= \frac{1}{\sigma^3} (\frac{6}{\lambda^3} - 3 \frac{1}{\lambda} \frac{2}{\lambda^2} + 3 \frac{1}{\lambda^2} \frac{1}{\lambda} - \frac{1}{\lambda^3})$$

$$= \lambda^3 (2/\lambda^3) = 2 \quad .$$

Therefore the value of γ_1 equals 2 for any Exponential distribution.

Also γ_2 is computing as follows:

$$\gamma_2 = E[\frac{(X-\mu)^4}{\sigma^4}] - 3$$

$$= E[X^4 - 4\mu X^3 + 6\mu^2 X^2 - 4\mu^3 X + \mu^4] / \sigma^4 - 3$$

(2) $$= \{E[X^4] - 4\mu E[X^3] + 6\mu^2 E[X^2] - 4\mu^3 E[X] + \mu^4\} / \sigma^4 - 3.$$

But

$$E[X^4] = \int_0^\infty x^4 \lambda e^{-\lambda x} \, dx$$

$$= \frac{\Gamma(5)}{\lambda^4} \int_0^\infty \frac{x^{5-1} \lambda^5 e^{-\lambda x}}{\Gamma(5)} \, dx$$

$$= 24 / \lambda^4 \quad .$$

Thus

$$\gamma_2 = \frac{1}{\sigma^4} \left\{ \frac{24}{\lambda^4} - 4 \frac{1}{\lambda} \frac{6}{\lambda^3} + 6 \frac{1}{\lambda^2} \frac{2}{\lambda^2} - 4 \frac{1}{\lambda^3} \frac{1}{\lambda} + \frac{1}{\lambda^4} \right\} - 3$$

$$= \lambda^4 \left(9 / \lambda^4 \right) - 3 = 6 \quad .$$

Therefore the value of γ_2 equals 6 for any Exponential distribution.

This fact is used by the Subroutine 'EXPLT' to give the user one more informal test for exponentiality. It estimates the values of γ_1 and γ_2 and if $\gamma_1 \neq 2$ and/or $\gamma_2 \neq 6$ then the data may not have an Exponential distribution.

Estimating the values of γ_1 and γ_2, 'EXPLT' uses unbiased estimator for σ, $E[(X-\mu)^3]$ and $E[(X-\mu)^4]$ using the formulas:

$$\sigma = \left(\sum_{i=1}^{n} (X_i - \bar{X})^2 / (n-1) \right)^{1/2} ,$$

$$E[(X-\mu)^3] = n \frac{\sum_{i=1}^{n} (X_i - \bar{X})^3}{(n-1)\ (n-2)} ,$$

$$E[(X-\mu)^4] = \frac{(n(n-2)+3) \left(\sum_{i=1}^{n} (X_i - \bar{X})^4 \right)}{(n-1)\ (n-2)\ (n-3)} - \frac{3 \sigma^4 (n-1)\ (2n-3)}{n\ (n-2)\ (n-3)} .$$

2. 'EXPLT' Structure

The Subroutine 'EXPLT' is also a FORTRAN callable subroutine with each call producing a plot for the given set of data and at the same time calculating and printing the estimated values of the parameters γ_1 and γ_2 of the data.

Basically the program is divided into two parts:

The first part is involved with all required computations calculating the Exponential Scores and the various statistics needed to compute the γ_1 and γ_2 parameters.

The second part is the Subroutine 'EPLOT'. 'EPLOT' scales the data points according to their range and plots the scaled data along 110 equal-spaced positions of the X-axis. No plot is given if the data have constant value or if there exists a data point less than zero.

A complete description of how 'EXPLT' operates is given by the Subroutine and a summary description is given on the terminal by typing the command DESCRIBE EXPLT under the CMS environment. When the user types DESCRIBE EXPLT the following response is printed on the terminal:

SUBROUTINE EXPLT

'EXPLT' takes a set of data, sorts it into increasing order and plots the created order statistics versus exponential scores (expected values of order statistics) computed by :

$$E[X_{(i)}] = \sum_{k=1}^{i} (i/(n-k+1)), \quad i = 1,2,\ldots,n .$$

Also, it computes the estimates of γ_1 and γ_2 parameters.

It is called by :

```
CALL   EXPLT  ( X, SCORES, N )
```

ARGUMENTS

X Is the array containing the data

SCORES Is a work array of dimension N

N Is the number of data values

More information is given in the subroutine.

3. Interpreting The Output

If data have an Exponential distribution then the plot will tend to a straight line. But it should be noted that the inverse is not generally true. The linearity of the graph is an indication only and gives the user a first feeling of the distribution of data. If the plot is not linear, however, this suggests that they are not Exponential. In addition the shape of the plot may lead the user to make an alternative model selection.

On the other hand the values of the statistics γ_1 and γ_2 may be used to test informally the distribution of data. These values are an indication only for the test for exponentiality. If the estimate of γ_1 has the value about 2 and the estimate of γ_2 has the value about 6 then it suggests that data may be exponentially distributed. But a departure of these values suggests that data do not have an exponential distribution.

4. Using 'EXPLT' with Data Generated from Various Distributions

The Subroutine 'EXPLT' has been used here, plotting data generated by the computer from various distributions, in order to get a visual sense of the behavior of the plot.

Figure 14a gives the plot of 100 Uniform (0,1) variates. Obviously nonlinearity governs the plot (as we should expect) and thus it suggests departure from the Exponential distribution. Besides the nonlinearity of the plot the estimates of γ_1 and γ_2 (0.1 and -1.15 respectively) are far away from the corresponding values (2 and 6) of exponentially distributed data. Figure 14b gives the associated histogram.

Figure 15a plots variates generated from a Triangular Symmetric distribution (0,2). The nonlinearity and the values of the estimates γ_1 and γ_2 suggest that one reject the exponential assumption.

Figure 16 shows the plot of X^2 generated variates with 10 degrees of freedom. For the same reasons as above the exponential model is rejected.

Figure 17a plots Normally (N(1000,1)) distributed generated data and the nonlinearity of the plot is apparent. Besides the nonlinearity, the estimates of γ_1 and γ_2 (.09 and -0.63) suggest not only to reject the Exponentiality of data but in addition lead us to test the data for Normality. Figure 17b gives the associated histogram.

Figure 18 plots exponentially generated variates with parameter $\lambda = 1$. As we see the graph is linear and the user gets the idea that the data possibly are exponential. Besides the linearity here, the values of the estimates of γ_1 and γ_2 (2.06 and 5.4) are close enough to their true values to indicate an Exponential model.

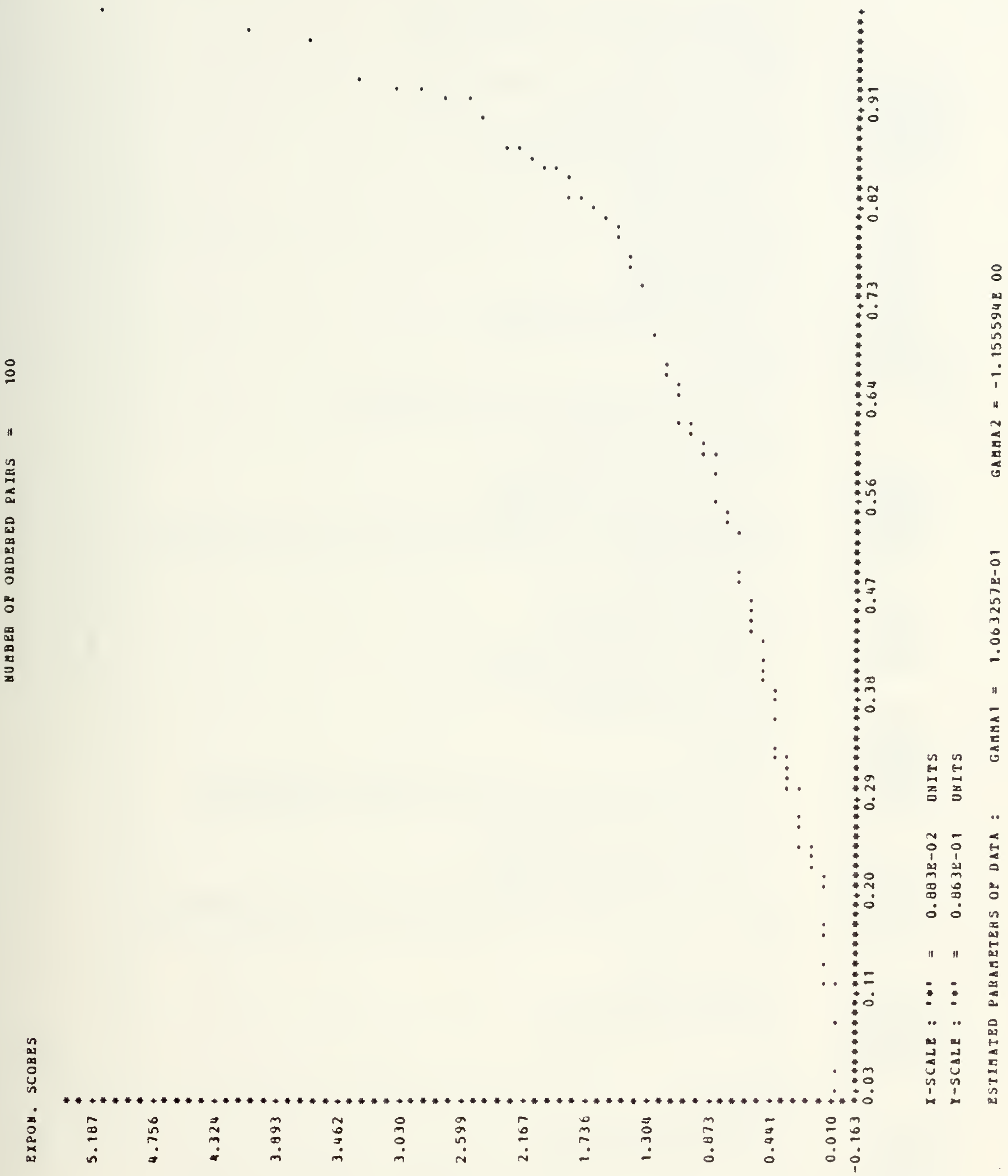
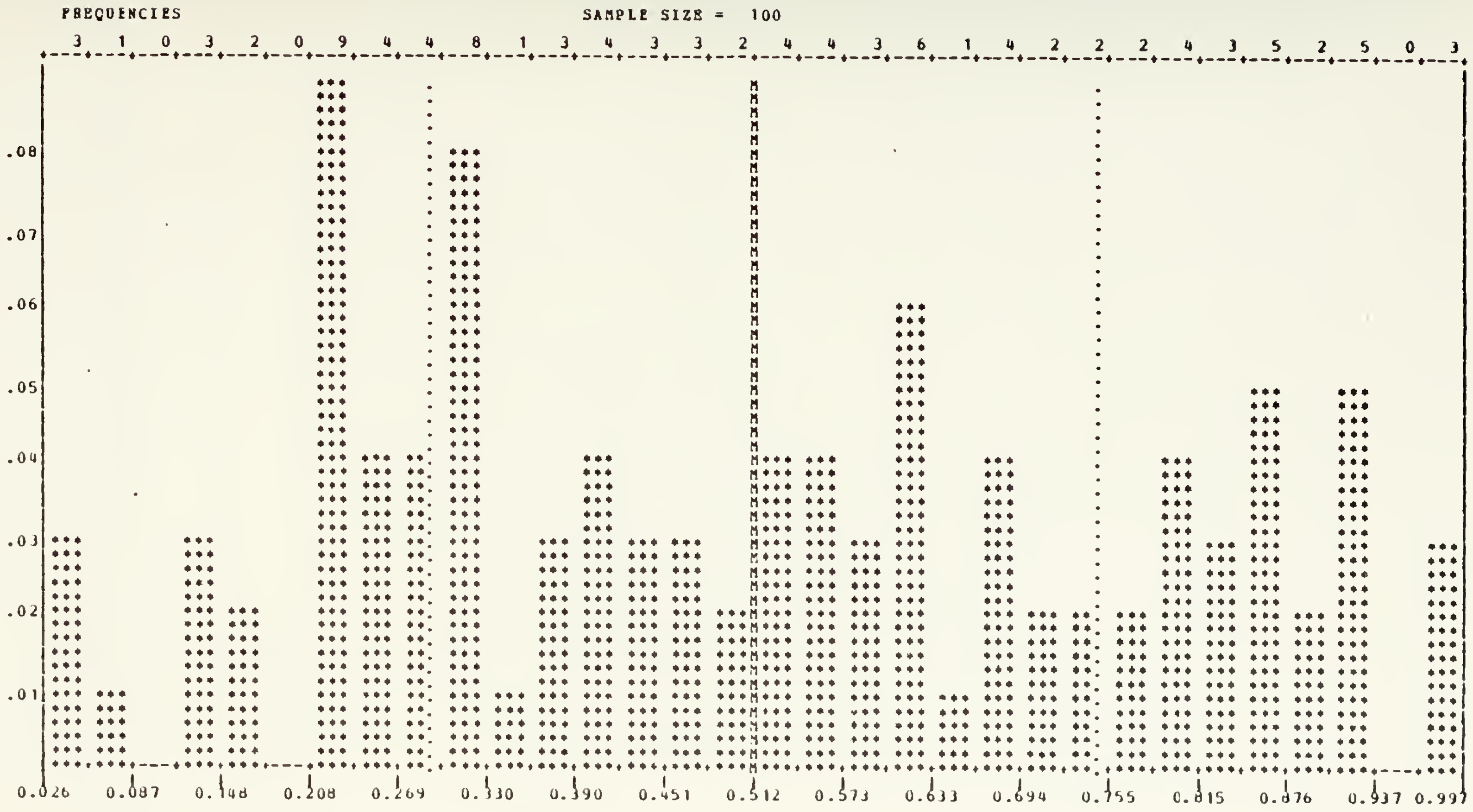
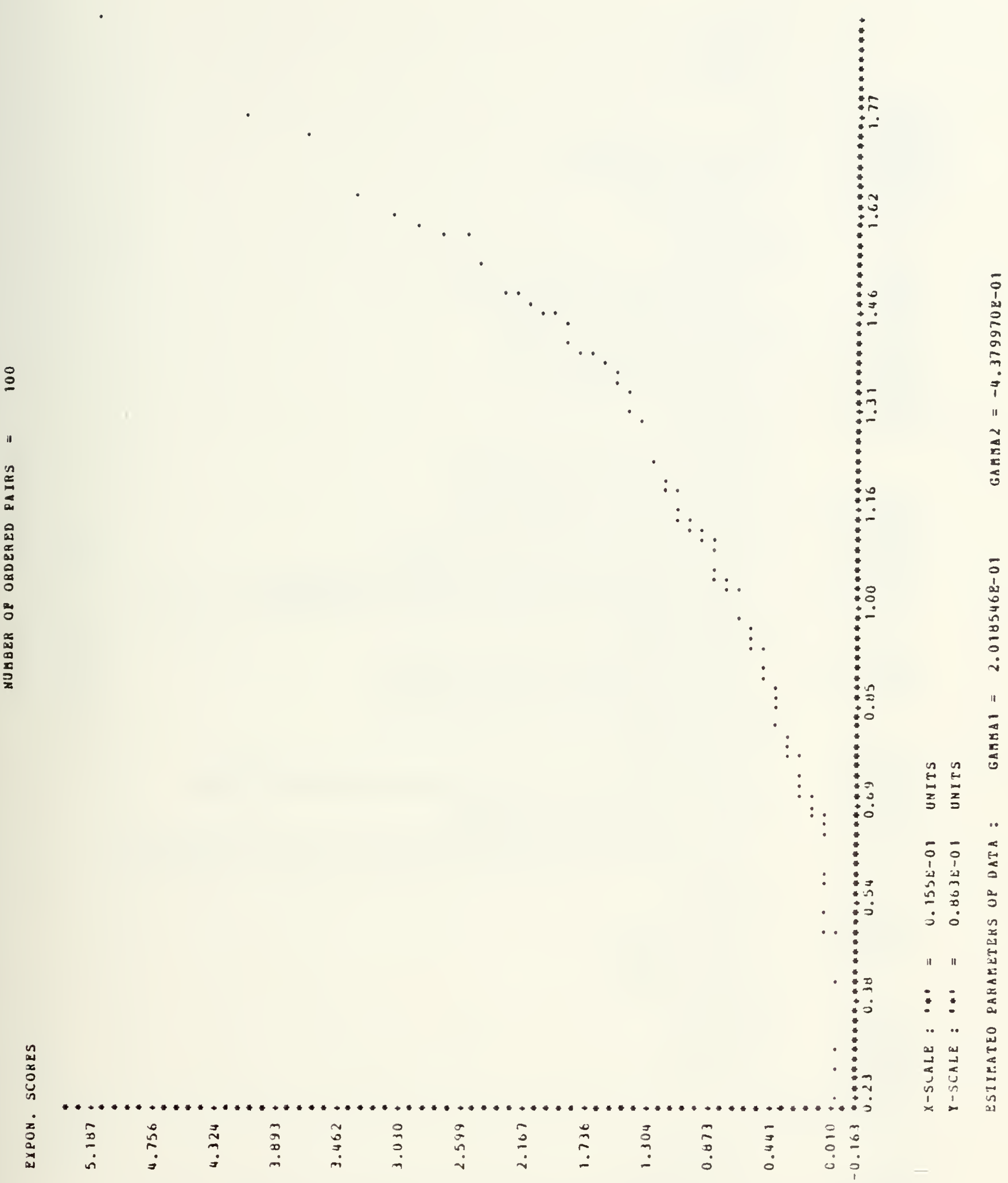


Figure 14a - A SAMPLE OF SIZE 100 FROM A UNIFORM(0,1) DISTRIBUTION



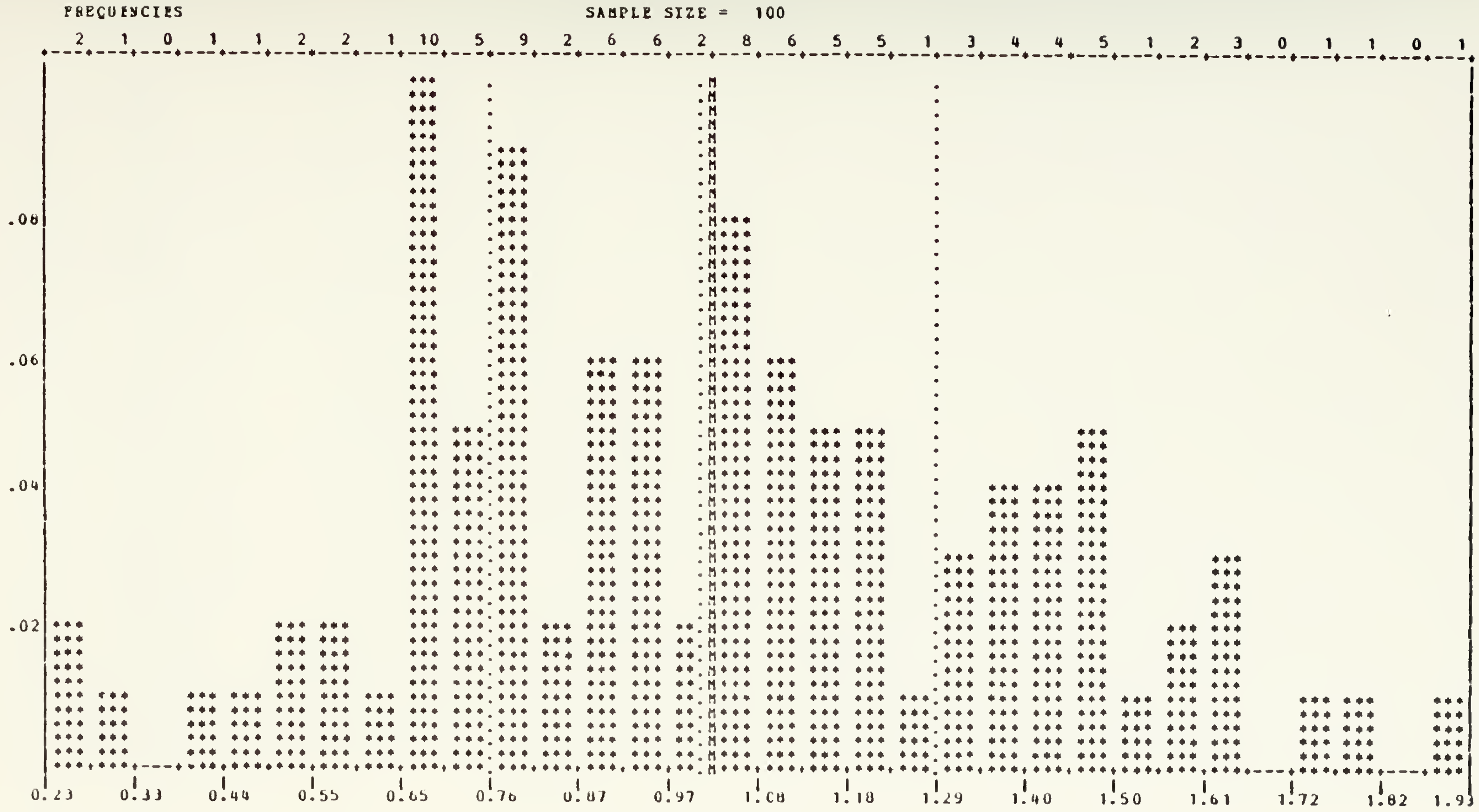
```
CENTRAL TENDENCY            SPREAD                      HIGHER CENTRAL MOMENTS      DISTRIBUTION

MEAN        5.130016E-01    VARIANCE   6.898522E-02     M3          1.926515E-03    MINIMUM                     2.616876E-02
MEDIAN      5.089474E-01    STD DEV    2.626504E-01     M4          8.777443E-03    .10 QUANTILE                2.121379E-01
TRIMEAN     5.136449E-01    COEF VAR   5.119874E-01     SKEWNESS    1.063258E-01    .25 QUANTILE  (HINGE)       2.858547E-01
MIDMEAN     4.978746E-01    MEAN DEV   2.265097E-01     KURTOSIS   -1.155596E 00    .50 QUANTILE  (MEDIAN)      5.089474E-01
MIDRANGE    5.118012E-01    RANGE      9.712653E-01     BETA1       1.869107E-03    .75 QUANTILE  (HINGE)       7.508311E-01
GEOM MEAN   4.255902E-01    MIDSPREAD  4.649764E-01     BETA2       8.707598E-03    .90 QUANTILE                8.750697E-01
HARM MEAN   2.965862E-01                                                            MAXIMUM                     9.974341E-01
```

Figure 14b - HISTOGRAM OF THE DATA OF FIGURE 14a



```
X-SCALE : *** = 0.155E-01 UNITS
Y-SCALE : *** = 0.863E-01 UNITS

ESTIMATED PARAMETERS OF DATA :     GAMMA1 = 2.018546E-01     GAMMA2 = -4.379970E-01
```

Figure 15a - A SAMPLE OF SIZE 100 FROM A TRIANGULAR SYMMETRIC (0,2) DISTRIBUTION



```
CENTRAL TENDENCY              SPREAD                          HIGHER CENTRAL MOMENTS       DISTRIBUTION

MEAN        1.024734E 00      VARIANCE    1.261122E-01        M3          9.040136E-03     MINIMUM                       2.287738E-01
MEDIAN      1.009097E 00      STD DEV     3.551228E-01        M4          4.074683E-02     .10 QUANTILE                  6.513643E-01
TRIMEAN     1.017067E 00      COEF VAR    3.465510E-01        SKEWNESS    2.018549E-01     .25 QUANTILE  (HINGE)         7.559967E-01
MIDMEAN     9.974750E-01      MEAN DEV    2.892413E-01        KURTOSIS   -4.379988E-01     .50 QUANTILE  (MEDIAN)        1.009097E 00
MIDRANGE    1.078568E 00      RANGE       1.699589E 00        BETA1       8.770742E-03     .75 QUANTILE  (HINGE)         1.294078E 00
GEOM MEAN   9.567293E-01      MIDSPREAD   5.380812E-01        BETA2       4.005696E-02     .90 QUANTILE                  1.500139E 00
HARM MEAN   8.757243E-01                                                                   MAXIMUM                       1.928363E 00
```

Figure 15b - HISTOGRAM OF THE DATA OF FIGURE 15a

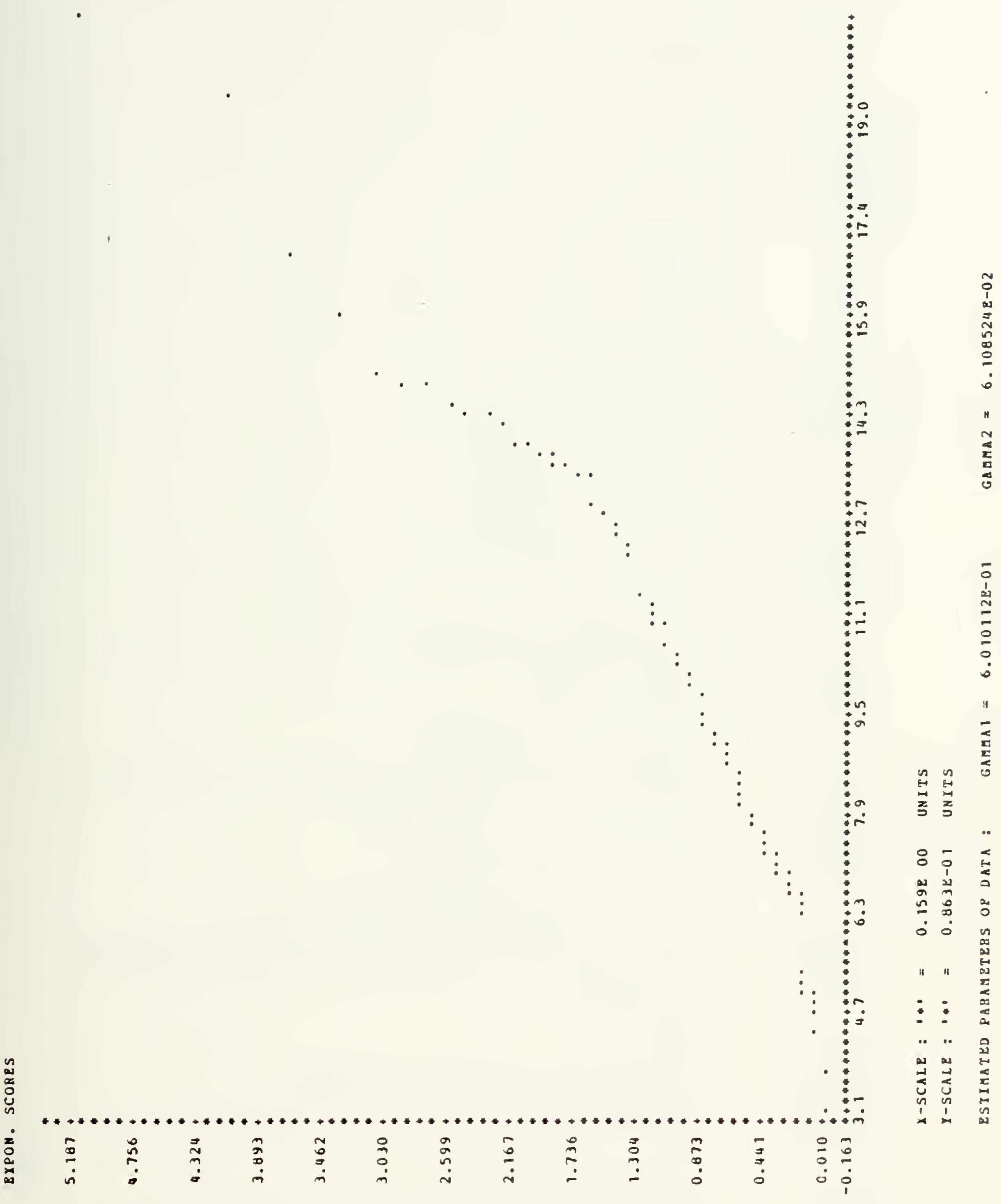
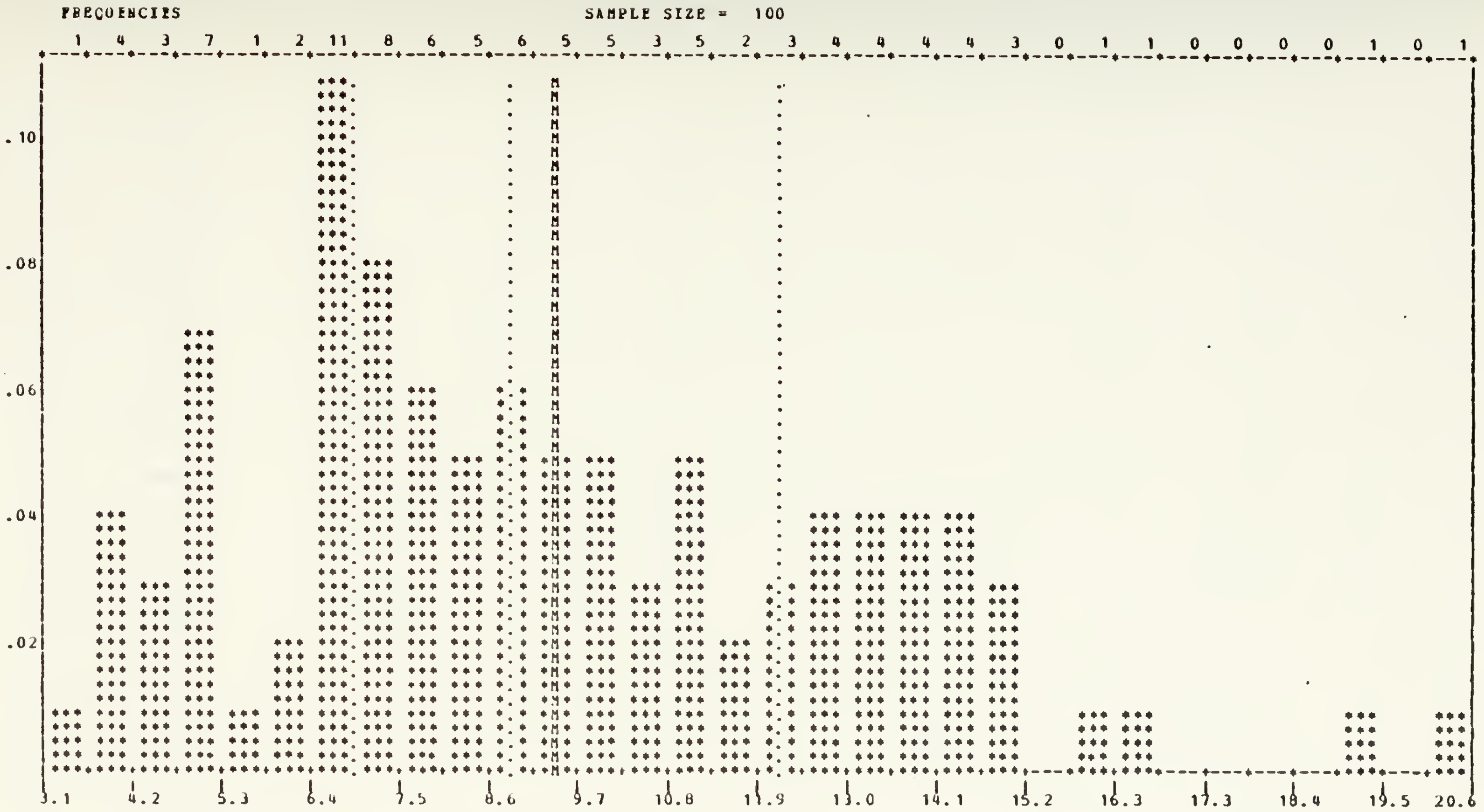
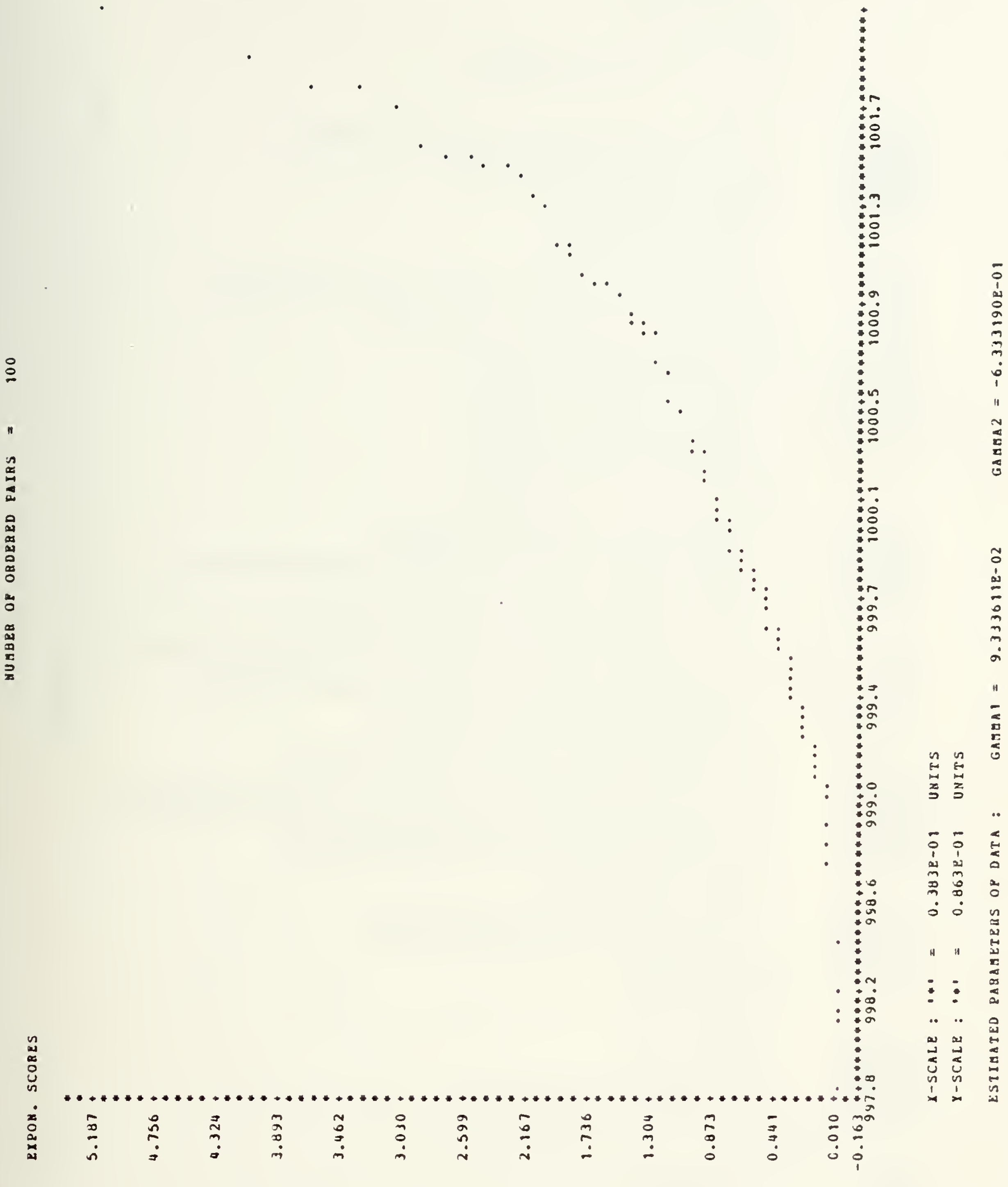


Figure 16a - X^2 (10 d.f.) DATA



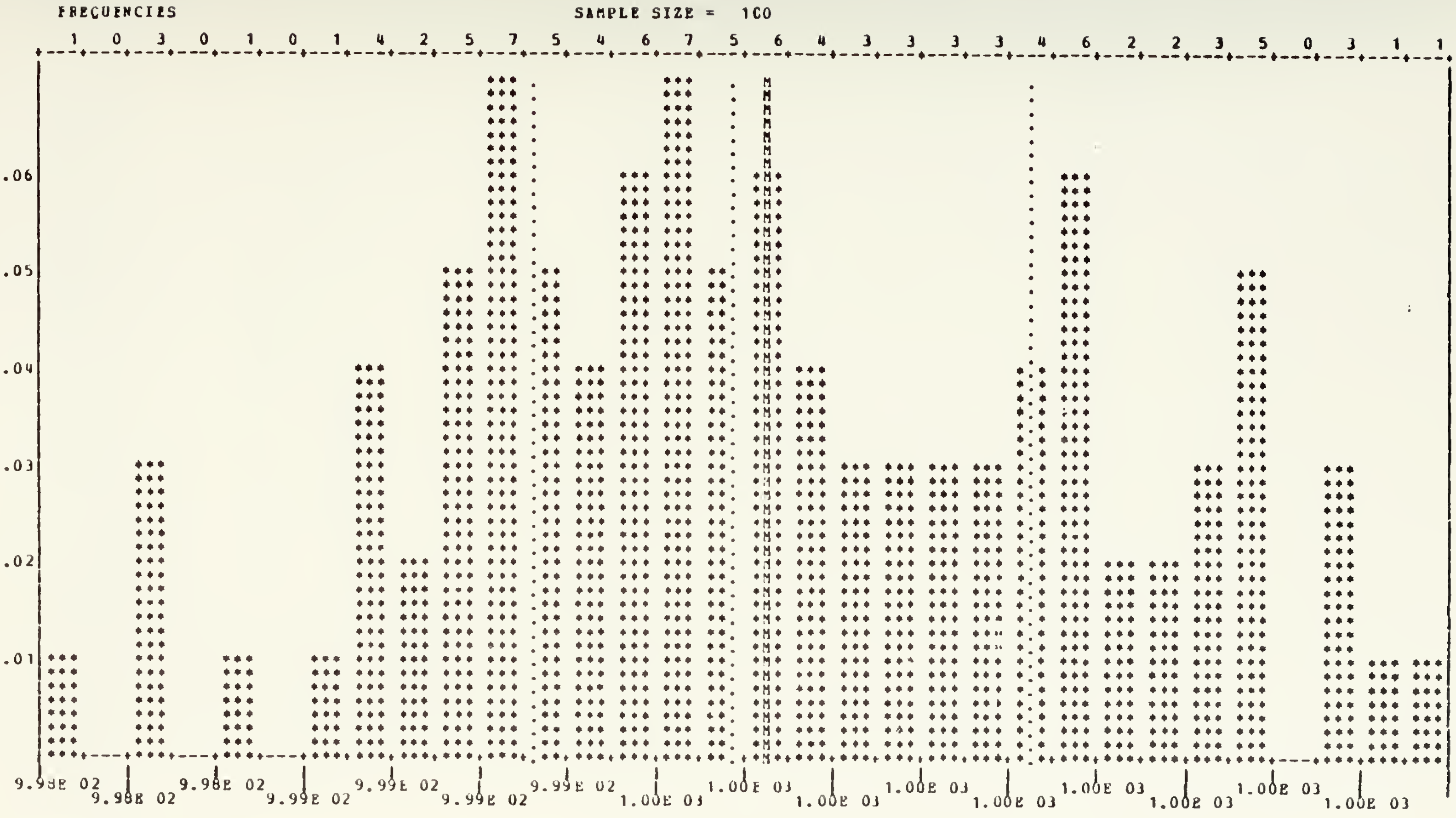
```
CENTRAL TENDENCY            SPREAD                      HIGHER CENTRAL MOMENTS      DISTRIBUTION

MEAN       9.338476E 00     VARIANCE   1.295645E 01     M3         2.802921E 01     MINIMUM                       3.126442E 00
MEDIAN     8.851341E 00     STD DEV    3.599506E 00     M4         5.138620E 02     .10 QUANTILE                  4.848789E 00
TRIMEAN    9.180511E 00     COEF VAR   3.854490E-01     SKEWNESS   6.010105E-01     .25 QUANTILE (HINGE)          6.910625E 00
MIDMEAN    8.899713E 00     MEAN DEV   2.890594E 00     KURTOSIS   6.106379E-02     .50 QUANTILE (MEDIAN)         8.851341E 00
MIDRANGE   1.187917E 01     RANGE      1.750546E 01     BETA1      2.719394E 01     .75 QUANTILE (HINGE)          1.210875E 01
GEOM MEAN  8.652583E 00     MIDSPREAD  5.198124E 00     BETA2      5.032305E 02     .90 QUANTILE                  1.416343E 01
HARM MEAN  7.969136E 00                                                             MAXIMUM                       2.063191E 01
```

Figure 16b - HISTOGRAM OF THE DATA OF FIGURE 16a



```
X-SCALE : '*' = 0.383E-01 UNITS
Y-SCALE : '*' = 0.863E-01 UNITS

ESTIMATED PARAMETERS OF DATA :     GAMMA1 = 9.333611E-02     GAMMA2 = -6.333190E-01
```

Figure 17a - A SAMPLE OF SIZE 100 FROM A NORMAL (1000,1) DISTRIBUTION



```
CENTRAL TENDENCY              SPREAD                        HIGHER CENTRAL MOMENTS     DISTRIBUTION

MEAN         1.000012E 03     VARIANCE    8.885761E-01      M3           7.817924E-02  MINIMUM                       9.978228E 02
MEDIAN       9.999041E 02     STD DEV     9.426432E-01      M4           1.866653E 00  .10 QUANTILE                  9.988616E 02
TRIMEAN      9.999739E 02     COEF VAR    9.426321E-04      SKEWNESS     9.333605E-02  .25 QUANTILE (HINGE)          9.992959E 02
MIDMEAN      9.999553E 02     MEAN DEV    7.691577E-01      KURTOSIS    -6.333208E-01  .50 QUANTILE (MEDIAN)         9.999041E 02
MIDRANGE     9.999314E 02     RANGE       4.217285E 00      BETA1        7.584953E-02  .75 QUANTILE (HINGE)          1.000792E 03
GEOM MEAN    1.000011E 03     MIDSPREAD   1.495605E 00      BETA2        1.840573E 00  .90 QUANTILE                  1.001375E 03
HARM MEAN    1.000011E 03                                                              MAXIMUM                       1.002040E 03
```

Figure 17b - HISTOGRAM OF THE DATA OF FIGURE 17a

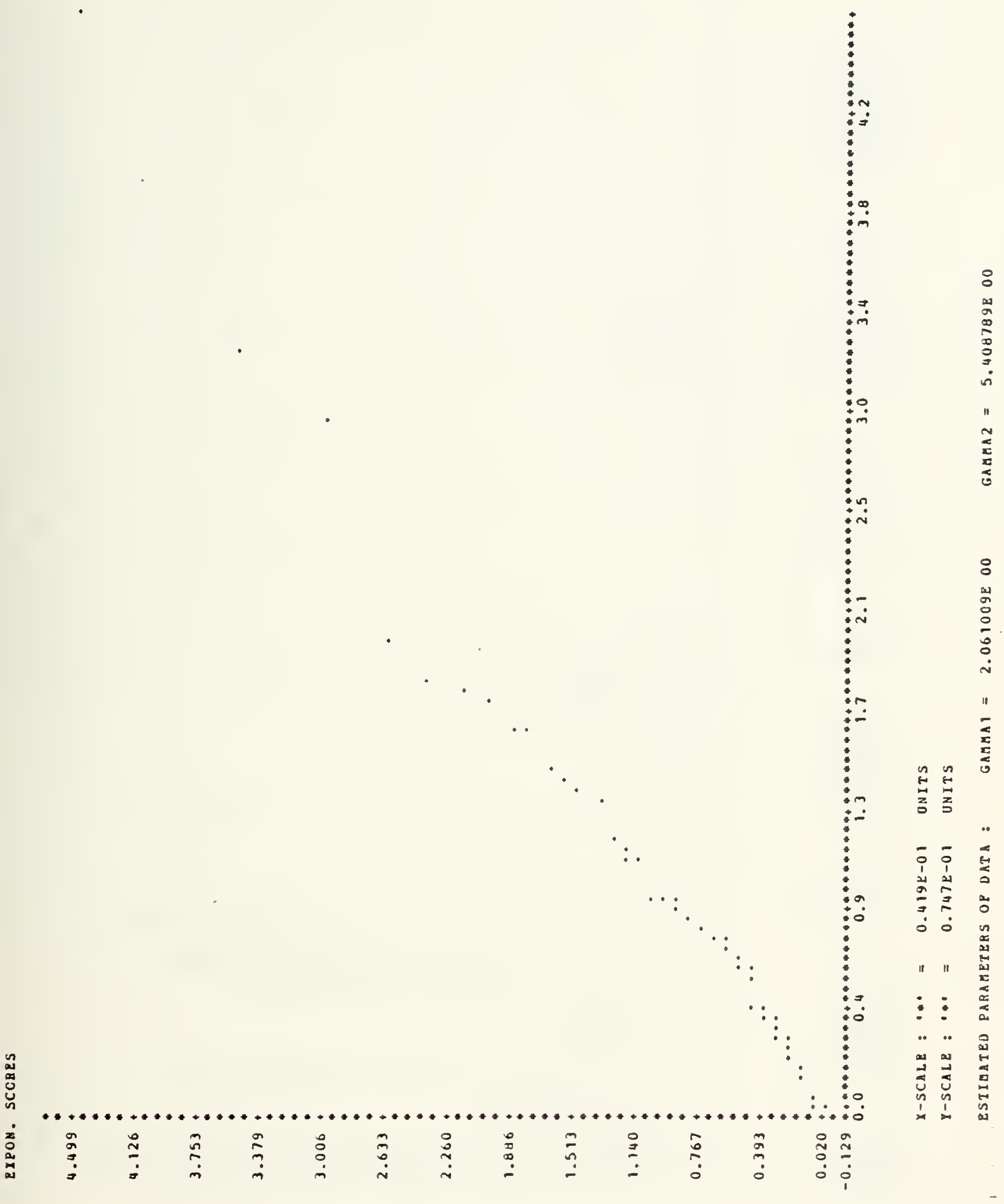
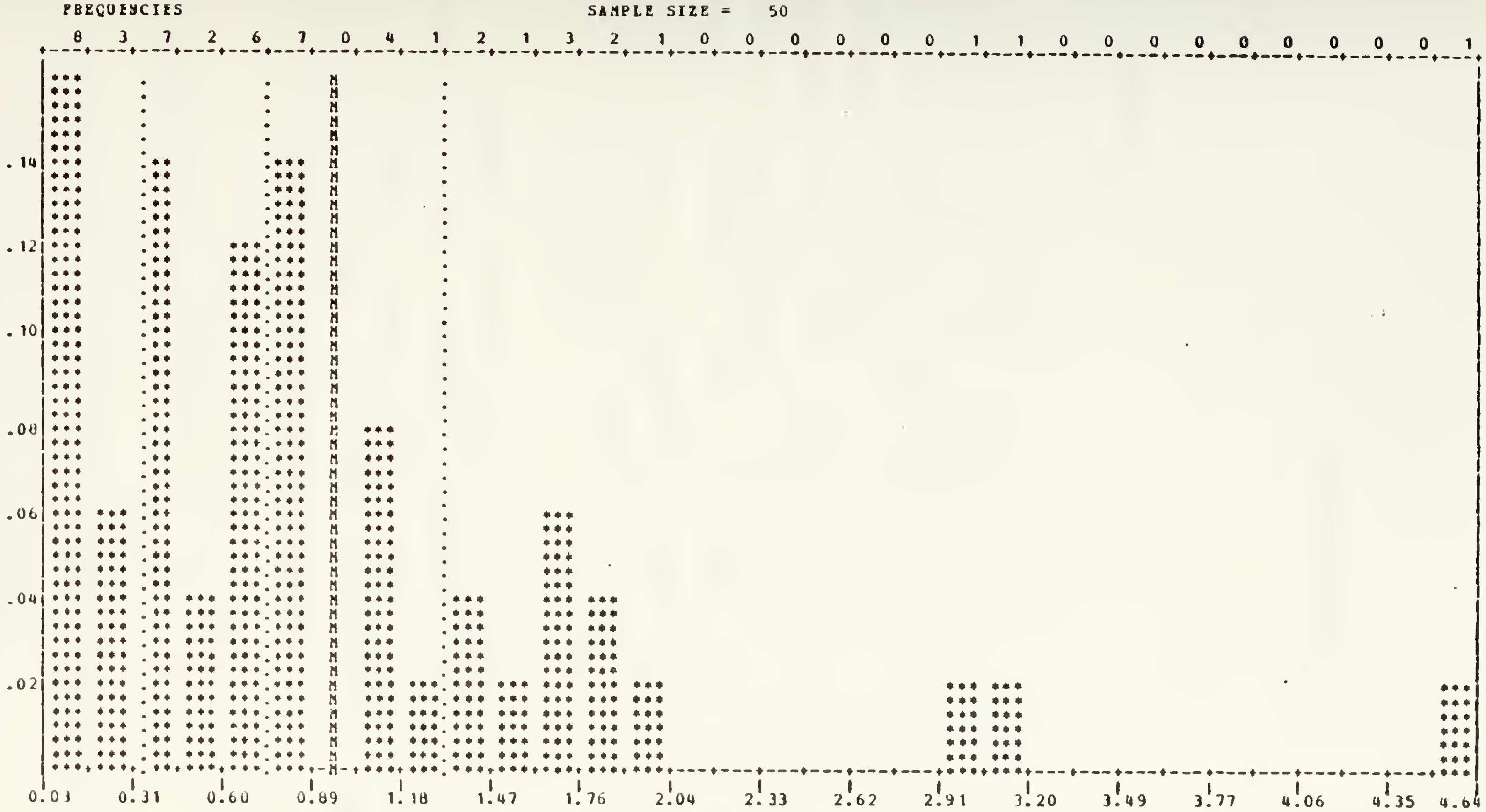


Figure 18a - A SAMPLE OF SIZE 50 FROM AN EXPONENTIAL DISTRIBUTION



```
CENTRAL TENDENCY          SPREAD                      HIGHER CENTRAL MOMENTS     DISTRIBUTION

MEAN       9.318499E-01   VARIANCE   7.832906E-01     M3        1.428776E 00     MINIMUM                    2.607422E-02
MEDIAN     7.331042E-01   STD DEV    8.850371E-01     M4        5.159163E 00     .10 QUANTILE               6.173313E-02
TRIMEAN    7.770853E-01   COEF VAR   9.497635E-01     SKEWNESS  2.061007E 00     .25 QUANTILE (HINGE)       3.298096E-01
MIDMEAN    7.460496E-01   MEAN DEV   5.913107E-01     KURTOSIS  5.408787E 00     .50 QUANTILE (MEDIAN)      7.331042E-01
MIDRANGE   2.333037E 00   RANGE      4.613927E 00     BETA1     1.344193E 00     .75 QUANTILE (HINGE)       1.312325E 00
GEOM MEAN  5.488822E-01   MIDSPREAD  9.825149E-01     BETA2     4.818045E 00     .90 QUANTILE               1.772292E 00
HARM MEAN  2.060167E-01                                                          MAXIMUM                    4.640001E 00
```

Figure 18b - HISTOGRAM OF THE DATA OF FIGURE 18a

IV. SUBROUTINE LIST

A. DESCRIPTION

Another Subroutine presented in this thesis is the Subroutine 'LIST'. Its purpose is to list a given set of data in ascending order and, taking advantage of like occurrences in the data, to print the ordered data in a compressed form. This feature becomes highly useful when listing a large number of data points that contain many repeated values. It is also a tool for finding multiplicities in supposedly continuous data, and a probability function estimating routine for data which is known to be discrete.

A complete description of how 'LIST' operates is given in the Subroutine. However a summary is given by typing DESCRIBE LIST. When the user types the command DESCRIBE LIST under the CMS environment the following response will be given on the terminal:

```
    SUBROUTINE  LIST

    'LIST' sorts a set of data  into  increasing  order  and
gives a 5-column print-out as follows:
   1st  column:  Serial  number of the first  occurrence  of
this value in the ordered List
   2nd  column:  Value of ordered data-value
   3rd  column:  Frequency  of occurrence of the value
```

4th column: Percent for the value

5th column: Graphical representation of the frequency for each value.

It is called by:

```
CALL   LIST  ( X, N )
```

where:

X Is the array of data

N Is the number of data-values.

More information is given in the subroutine.

E. INTERPRETING THE OUTPUT

The print-out is a visual representation of the data and of each data value frequency. For data points having the same value 'LIST' will print this value once with the number of occurrences. A 5-column output will be printed and its interpretation is:

1. First column gives the serial number of the first occurrence of this value in the ordered List
2. Second column gives the value of ordered data-value
3. Third column gives the frequency of occurrence of the value
4. Forth column gives the percent for the value
5. Fifth column is a graphical representation of the frequency for each value.

Example:

Let 1, 1, 5, 3, 5, 1, 2, 2, 5, 6 be a given data set. Then the ordered data will be: $X_{(1)} = X_{(2)} = X_{(3)} = 1$, $X_{(4)} = X_{(5)} = 2$, $X_{(6)} = 3$, $X_{(7)} = X_{(8)} = X_{(9)} = 5$, $X_{(10)} = 6$. Then the print-out will have the following form:

```
SERIAL NUMBER  ORDERED DATA  FREQUENCY  PERCENT  PROB. GRAPH

      1             1            3         .3         ***
      4             2            2         .2         **
      6             3            1         .1         *
      7             5            3         .3         ***
     10             6            1         .1         *
```

If there are no data-points having a common value then 'LIST' gives this indication and prints only the ordered data. This happens when data have a continuous distribution.

C. USING 'LIST' WITH TELEPHONE DATA 1 AND TELEPHONE DATA 2

Subroutine 'LIST' was used with Telephone Data 1 (Figure 19) and Telephone Data 2 (Figure 20) and a brief analysis of the output follows.

Looking at Figure 19 and Figure 20 we may get some information of each data set. It can be seen that both of the data sets contain a large number of multiple occurrences of the data value one and data value two. As we can see reading the fourth column the occurrence of ones is 19% for Telephone Data 1 and 24% for Telephone Data 2.

Comparing also Figures 19a-c, 20a-d we can see that multiple occurrences happen in the range 1 to 240 for Telephone Data 1 and in the range 1 to 132 for Telephone Data 2.

Furthermore, quick visual information concerning the range where we have multiple occurrences can be obtained from the Probability Graph. Thus we can see that for Telephone Data 2 there is a region from 113 to 132 where the multiple occurrences of values is larger than in neighboring regions. Therefore 'LIST' gives the user a useful tool for analysis of data.

```
SERIAL NUMBER      ORDERED DATA       FREQUENCIES      PERCENT               PROBABILITY  GRAPH

      1         0.10000000E 01          128          0.19048      ****************************************
    129         0.20000000E 01           54          0.08036      *****************
    183         0.30000000E 01           28          0.04167      *********
    211         0.40000000E 01           22          0.03274      *******
    233         0.50000000E 01           17          0.02530      *****
    250         0.60000000E 01           11          0.01637      ***
    261         0.70000000E 01           10          0.01488      ***
    271         0.80000000E 01           12          0.01786      ****
    283         0.90000000E 01           14          0.02083      ****
    297         0.10000000E 02            9          0.01339      ***
    306         0.11000000E 02           10          0.01488      ***
    316         0.12000000E 02           11          0.01637      ***
    327         0.13000000E 02            6          0.00893      **
    333         0.14000000E 02            6          0.00893      **
    339         0.15000000E 02            6          0.00893      **
    345         0.16000000E 02            8          0.01190      ***
    353         0.17000000E 02            8          0.01190      ***
    361         0.18000000E 02            5          0.00744      **
    366         0.19000000E 02           12          0.01786      ****
    378         0.20000000E 02            1          0.00149
    379         0.21000000E 02            5          0.00744      **
    384         0.22000000E 02            5          0.00744      **
    389         0.23000000E 02            3          0.00446      *
    392         0.24000000E 02            7          0.01042      **
    399         0.25000000E 02            3          0.00446      *
    402         0.26000000E 02            3          0.00446      *
    405         0.27000000E 02            2          0.00298      *
    407         0.28000000E 02            3          0.00446      *
    410         0.29000000E 02            5          0.00744      **
    415         0.30000000E 02            6          0.00893      **
    421         0.31000000E 02            4          0.00595      *
    425         0.32000000E 02            4          0.00595      *
    429         0.33000000E 02            2          0.00298      *
    431         0.34000000E 02            4          0.00595      *
    435         0.35000000E 02            3          0.00446      *
    438         0.37000000E 02            2          0.00298      *
    440         0.38000000E 02            2          0.00298      *
    442         0.39000000E 02            1          0.00149
    443         0.40000000E 02            2          0.00298      *
    445         0.41000000E 02            2          0.00298      *
    447         0.43000000E 02            1          0.00149
    448         0.44000000E 02            1          0.00149
    449         0.45000000E 02            4          0.00595      *
    453         0.46000000E 02            3          0.00446      *
    456         0.47000000E 02            1          0.00149
    457         0.48000000E 02            2          0.00298      *
    459         0.49000000E 02            1          0.00149
    460         0.50000000E 02            1          0.00149
    461         0.51000000E 02            2          0.00298      *
    463         0.52000000E 02            1          0.00149
    464         0.53000000E 02            1          0.00149
    465         0.54000000E 02            3          0.00446      *
    468         0.55000000E 02            2          0.00298      *
    470         0.56000000E 02            2          0.00298      *
    472         0.58000000E 02            2          0.00298      *
    474         0.59000000E 02            1          0.00149
    475         0.62000000E 02            2          0.00298      *
    477         0.63000000E 02            3          0.00446      *
    480         0.64000000E 02            1          0.00149
    481         0.65000000E 02            2          0.00298      *
    483         0.66000000E 02            1          0.00149
    484         0.68000000E 02            3          0.00446      *
    487         0.69000000E 02            1          0.00149
    488         0.70000000E 02            1          0.00149
    489         0.73000000E 02            1          0.00149
    490         0.74000000E 02            2          0.00298      *
    492         0.75000000E 02            1          0.00149
    493         0.79000000E 02            3          0.00446      *
    496         0.83000000E 02            1          0.00149
    497         0.84000000E 02            1          0.00149
    498         0.86000000E 02            1          0.00149
    499         0.88000000E 02            2          0.00298      *
    501         0.89000000E 02            1          0.00149
    502         0.90000000E 02            2          0.00298      *
    504         0.91000000E 02            1          0.00149
    505         0.93000000E 02            1          0.00149
    506         0.95000000E 02            1          0.00149
```

Figure 19a - USING LIST WITH TELEPHONE DATA SET 1.

```
507          0.98000000E 02          1          0.00149
508          0.99000000E 02          1          0.00149
509          0.10600000E 03          1          0.00149
510          0.10900000E 03          1          0.00149
511          0.11100000E 03          1          0.00149
512          0.11200000E 03          2          0.00298      *
514          0.11300000E 03          1          0.00149
515          0.11600000E 03          1          0.00149
516          0.11700000E 03          1          0.00149
517          0.11900000E 03          1          0.00149
518          0.12000000E 03          4          0.00595      *
522          0.12100000E 03          2          0.00298      *
524          0.12200000E 03          2          0.00298      *
526          0.12300000E 03          4          0.00595      *
530          0.12400000E 03          3          0.00446      *
533          0.12800000E 03          1          0.00149
534          0.13500000E 03          1          0.00149
535          0.14200000E 03          1          0.00149
536          0.14800000E 03          1          0.00149
537          0.15300000E 03          1          0.00149
538          0.15600000E 03          1          0.00149
539          0.15800000E 03          1          0.00149
540          0.16100000E 03          1          0.00149
541          0.16500000E 03          1          0.00149
542          0.17500000E 03          1          0.00149
543          0.17600000E 03          1          0.00149
544          0.17700000E 03          1          0.00149
545          0.18300000E 03          1          0.00149
546          0.18600000E 03          1          0.00149
547          0.18700000E 03          1          0.00149
548          0.19200000E 03          1          0.00149
549          0.19300000E 03          1          0.00149
550          0.20200000E 03          1          0.00149
551          0.21700000E 03          2          0.00298      *
553          0.22400000E 03          1          0.00149
554          0.22600000E 03          1          0.00149
555          0.22800000E 03          3          0.00446      *
558          0.23100000E 03          1          0.00149
559          0.23400000E 03          1          0.00149
560          0.23700000E 03          1          0.00149
561          0.23900000E 03          2          0.00298      *
563          0.24000000E 03          4          0.00595      *
567          0.24100000E 03          1          0.00149
568          0.24400000E 03          1          0.00149
569          0.24800000E 03          1          0.00149
570          0.24900000E 03          1          0.00149
571          0.25100000E 03          1          0.00149
572          0.25200000E 03          1          0.00149
573          0.27000000E 03          1          0.00149
574          0.27900000E 03          1          0.00149
575          0.28600000E 03          1          0.00149
576          0.29700000E 03          1          0.00149
577          0.30300000E 03          1          0.00149
578          0.31200000E 03          1          0.00149
579          0.31800000E 03          1          0.00149
580          0.34700000E 03          1          0.00149
581          0.35400000E 03          1          0.00149
582          0.36000000E 03          1          0.00149
583          0.36400000E 03          1          0.00149
584          0.37000000E 03          1          0.00149
585          0.39000000E 03          1          0.00149
586          0.39400000E 03          1          0.00149
587          0.43500000E 03          1          0.00149
588          0.46000000E 03          1          0.00149
589          0.46500000E 03          1          0.00149
590          0.47300000E 03          1          0.00149
591          0.48000000E 03          1          0.00149
592          0.49100000E 03          1          0.00149
593          0.54900000E 03          1          0.00149
594          0.60000000E 03          2          0.00298      *
596          0.60900000E 03          1          0.00149
597          0.62100000E 03          1          0.00149
598          0.71100000E 03          1          0.00149
599          0.81700000E 03          1          0.00149
600          0.83600000E 03          1          0.00149
601          0.92700000E 03          1          0.00149
602          0.11240000E 04          1          0.00149
603          0.11500000E 04          1          0.00149
604          0.12700000E 04          1          0.00149
605          0.12890000E 04          1          0.00149
```

Figure 19b - USING LIST WITH TELEPHONE DATA SET 1
(cont.)

```
606           0.12980000E 04          1          0.00149
607           0.13050000E 04          1          0.00149
608           0.13280000E 04          1          0.00149
609           0.13480000E 04          1          0.00149
610           0.13550000E 04          1          0.00149
611           0.14120000E 04          1          0.00149
612           0.14290000E 04          1          0.00149
613           0.14890000E 04          1          0.00149
614           0.14930000E 04          1          0.00149
615           0.15100000E 04          1          0.00149
616           0.15190000E 04          1          0.00149
617           0.15470000E 04          1          0.00149
618           0.16330000E 04          1          0.00149
619           0.17870000E 04          1          0.00149
620           0.20720000E 04          1          0.00149
621           0.24830000E 04          1          0.00149
622           0.28060000E 04          1          0.00149
623           0.29620000E 04          1          0.00149
624           0.30260000E 04          1          0.00149
625           0.32010000E 04          1          0.00149
626           0.35930000E 04          1          0.00149
627           0.36850000E 04          1          0.00149
628           0.39520000E 04          1          0.00149
629           0.41570000E 04          1          0.00149
630           0.44690000E 04          1          0.00149
631           0.62080000E 04          1          0.00149
632           0.76140000E 04          1          0.00149
633           0.83220000E 04          1          0.00149
634           0.90150000E 04          1          0.00149
635           0.96250000E 04          1          0.00149
636           0.98060000E 04          1          0.00149
637           0.98180000E 04          1          0.00149
638           0.10154000E 05          1          0.00149
639           0.10398000E 05          1          0.00149
640           0.10451000E 05          1          0.00149
641           0.10939000E 05          1          0.00149
642           0.11280000E 05          1          0.00149
643           0.13447000E 05          1          0.00149
644           0.14385000E 05          1          0.00149
645           0.15155000E 05          1          0.00149
646           0.15294000E 05          1          0.00149
647           0.15504000E 05          1          0.00149
648           0.15847000E 05          1          0.00149
649           0.15868000E 05          1          0.00149
650           0.16280000E 05          1          0.00149
651           0.16299000E 05          1          0.00149
652           0.16361000E 05          1          0.00149
653           0.16408000E 05          1          0.00149
654           0.16817000E 05          1          0.00149
655           0.17174000E 05          1          0.00149
656           0.17667000E 05          1          0.00149
657           0.18218000E 05          1          0.00149
658           0.18649000E 05          1          0.00149
659           0.19461000E 05          1          0.00149
660           0.21848000E 05          1          0.00149
661           0.23498000E 05          1          0.00149
662           0.24692000E 05          1          0.00149
663           0.26443000E 05          1          0.00149
664           0.30974000E 05          1          0.00149
665           0.35644000E 05          1          0.00149
666           0.38003000E 05          1          0.00149
667           0.40131000E 05          1          0.00149
668           0.47120000E 05          1          0.00149
669           0.47592000E 05          1          0.00149
670           0.61710000E 05          1          0.00149
671           0.69775000E 05          1          0.00149
672           0.85993000E 05          1          0.00149
```

Figure 19c -   USING LIST WITH TELEPHONE DATA SET 1
(cont.)

```
SERIAL NUMBER    ORDERED DATA   FREQUENCIES       PERCENT        PROBABILITY GRAPH

         1      0.10000000E 01         178        0.24185        ****************************************
       179      0.20000000E 01          36        0.04891        ********
       215      0.30000000E 01          11        0.01495        **
       226      0.40000000E 01           6        0.00815        *
       232      0.50000000E 01           6        0.00815        *
       238      0.60000000E 01           5        0.00679        *
       243      0.70000000E 01           5        0.00679        *
       248      0.80000000E 01           4        0.00543        *
       252      0.90000000E 01           4        0.00543        *
       256      0.10000000E 02           9        0.01223        **
       265      0.11000000E 02           2        0.00272
       267      0.12000000E 02           3        0.00408        *
       270      0.13000000E 02           1        0.00136
       271      0.14000000E 02           1        0.00136
       272      0.15000000E 02           1        0.00136
       273      0.16000000E 02           4        0.00543        *
       277      0.19000000E 02           1        0.00136
       278      0.21000000E 02           1        0.00136
       279      0.22000000E 02           1        0.00136
       280      0.24000000E 02           3        0.00408        *
       283      0.25000000E 02           3        0.00408        *
       286      0.26000000E 02           2        0.00272
       288      0.27000000E 02           1        0.00136
       289      0.30000000E 02           3        0.00408        *
       292      0.32000000E 02           1        0.00136
       293      0.36000000E 02           1        0.00136
       294      0.40000000E 02           2        0.00272
       296      0.42000000E 02           1        0.00136
       297      0.44000000E 02           1        0.00136
       298      0.47000000E 02           4        0.00543        *
       302      0.48000000E 02           1        0.00136
       303      0.49000000E 02           9        0.01223        **
       312      0.50000000E 02           4        0.00543        *
       316      0.53000000E 02           3        0.00408        *
       319      0.55000000E 02           2        0.00272
       321      0.56000000E 02           4        0.00543        *
       325      0.57000000E 02           3        0.00408        *
       328      0.58000000E 02           1        0.00136
       329      0.59000000E 02           3        0.00408        *
       332      0.60000000E 02           2        0.00272
       334      0.62000000E 02           3        0.00408        *
       337      0.63000000E 02           1        0.00136
       338      0.64000000E 02           4        0.00543        *
       342      0.65000000E 02           3        0.00408        *
       345      0.66000000E 02           4        0.00543        *
       349      0.67000000E 02           1        0.00136
       350      0.68000000E 02           5        0.00679        *
       355      0.69000000E 02           3        0.00408        *
       358      0.70000000E 02           3        0.00408        *
       361      0.71000000E 02           5        0.00679        *
       366      0.73000000E 02           1        0.00136
       367      0.74000000E 02           1        0.00136
       368      0.75000000E 02           1        0.00136
       369      0.76000000E 02           1        0.00136
       370      0.77000000E 02           1        0.00136
       371      0.81000000E 02           1        0.00136
       372      0.86000000E 02           1        0.00136
       373      0.88000000E 02           1        0.00136
       374      0.89000000E 02           1        0.00136
       375      0.93000000E 02           1        0.00136
       376      0.95000000E 02           1        0.00136
       377      0.96000000E 02           1        0.00136
       378      0.97000000E 02           1        0.00136
       379      0.99000000E 02           3        0.00408        *
       382      0.10000000E 03           1        0.00136
       383      0.10100000E 03           1        0.00136
       384      0.10500000E 03           1        0.00136
       385      0.10600000E 03           1        0.00136
       386      0.10700000E 03           1        0.00136
       387      0.10900000E 03           1        0.00136
       388      0.11000000E 03           1        0.00136
       389      0.11100000E 03           1        0.00136
       390      0.11200000E 03           1        0.00136
       391      0.11300000E 03           4        0.00543        *
       395      0.11400000E 03           3        0.00408        *
       398      0.11500000E 03           1        0.00136
       399      0.11600000E 03           4        0.00543        *
       403      0.11700000E 03           1        0.00136
```

Figure 20a - USING LIST WITH TELEPHONE DATA SET 2.

```
404          0.11800000E 03        5        0.00679      *
409          0.11900000E 03        8        0.01087      **
417          0.12000000E 03        9        0.01223      **
426          0.12100000E 03        7        0.00951      **
433          0.12200000E 03       12        0.01630      ***
445          0.12300000E 03        8        0.01087      **
453          0.12400000E 03        7        0.00951      **
460          0.12500000E 03        9        0.01223      **
469          0.12600000E 03        6        0.00815      *
475          0.12700000E 03        3        0.00408      *
478          0.12800000E 03        2        0.00272
480          0.12900000E 03        4        0.00543      *
484          0.13000000E 03        3        0.00408      *
487          0.13200000E 03        3        0.00408      *
490          0.13300000E 03        2        0.00272
492          0.13700000E 03        1        0.00136
493          0.14000000E 03        1        0.00136
494          0.14300000E 03        1        0.00136
495          0.15200000E 03        1        0.00136
496          0.15800000E 03        1        0.00136
497          0.16600000E 03        1        0.00136
498          0.16900000E 03        2        0.00272
500          0.17000000E 03        1        0.00136
501          0.17300000E 03        2        0.00272
503          0.17600000E 03        1        0.00136
504          0.17800000E 03        1        0.00136
505          0.18000000E 03        2        0.00272
507          0.18200000E 03        1        0.00136
508          0.18500000E 03        1        0.00136
509          0.18700000E 03        1        0.00136
510          0.19000000E 03        1        0.00136
511          0.19400000E 03        1        0.00136
512          0.19500000E 03        1        0.00136
513          0.19900000E 03        1        0.00136
514          0.20600000E 03        1        0.00136
515          0.20900000E 03        1        0.00136
516          0.21600000E 03        1        0.00136
517          0.22900000E 03        1        0.00136
518          0.23000000E 03        1        0.00136
519          0.23700000E 03        1        0.00136
520          0.23900000E 03        3        0.00408      *
523          0.24000000E 03        2        0.00272
525          0.24100000E 03        1        0.00136
526          0.24400000E 03        2        0.00272
528          0.24700000E 03        2        0.00272
530          0.24800000E 03        2        0.00272
532          0.25000000E 03        1        0.00136
533          0.25100000E 03        1        0.00136
534          0.25400000E 03        2        0.00272
536          0.25500000E 03        2        0.00272
538          0.25600000E 03        1        0.00136
539          0.25800000E 03        1        0.00136
540          0.27200000E 03        1        0.00136
541          0.27500000E 03        1        0.00136
542          0.28000000E 03        1        0.00136
543          0.29800000E 03        1        0.00136
544          0.31800000E 03        1        0.00136
545          0.32400000E 03        1        0.00136
546          0.34000000E 03        1        0.00136
547          0.34400000E 03        1        0.00136
548          0.34600000E 03        1        0.00136
549          0.35900000E 03        1        0.00136
550          0.36500000E 03        2        0.00272
552          0.36700000E 03        1        0.00136
553          0.37700000E 03        1        0.00136
554          0.38000000E 03        1        0.00136
555          0.38700000E 03        1        0.00136
556          0.41300000E 03        1        0.00136
557          0.41800000E 03        1        0.00136
558          0.42900000E 03        1        0.00136
559          0.45300000E 03        1        0.00136
560          0.46600000E 03        1        0.00136
561          0.47200000E 03        1        0.00136
562          0.50300000E 03        2        0.00272
564          0.50400000E 03        1        0.00136
565          0.50700000E 03        1        0.00136
566          0.51000000E 03        1        0.00136
567          0.51400000E 03        1        0.00136
568          0.51700000E 03        1        0.00136
569          0.52800000E 03        1        0.00136
```

Figure 20b - USING LIST WITH TELEPHONE DATA SET 2
(cont.)

```
570              0.57000000E 03          1           0.00136
571              0.57200000E 03          1           0.00136
572              0.58600000E 03          1           0.00136
573              0.58700000E 03          1           0.00136
574              0.60100000E 03          1           0.00136
575              0.61400000E 03          1           0.00136
576              0.62400000E 03          1           0.00136
577              0.64000000E 03          1           0.00136
578              0.66300000E 03          1           0.00136
579              0.71400000E 03          1           0.00136
580              0.71600000E 03          2           0.00272
582              0.74000000E 03          1           0.00136
583              0.74200000E 03          2           0.00272
585              0.79100000E 03          1           0.00136
586              0.85300000E 03          1           0.00136
587              0.86200000E 03          1           0.00136
588              0.86600000E 03          1           0.00136
589              0.87500000E 03          1           0.00136
590              0.88500000E 03          1           0.00136
591              0.89400000E 03          1           0.00136
592              0.92600000E 03          1           0.00136
593              0.92800000E 03          1           0.00136
594              0.94100000E 03          1           0.00136
595              0.97600000E 03          1           0.00136
596              0.99100000E 03          1           0.00136
597              0.10210000E 04          1           0.00136
598              0.10230000E 04          1           0.00136
599              0.10620000E 04          1           0.00136
600              0.10860000E 04          2           0.00272
602              0.11070000E 04          1           0.00136
603              0.11190000E 04          1           0.00136
604              0.11250000E 04          1           0.00136
605              0.12310000E 04          1           0.00136
606              0.12500000E 04          1           0.00136
607              0.12610000E 04          2           0.00272
609              0.12960000E 04          1           0.00136
610              0.13670000E 04          1           0.00136
611              0.14120000E 04          1           0.00136
612              0.14130000E 04          1           0.00136
613              0.14620000E 04          1           0.00136
614              0.14850000E 04          1           0.00136
615              0.14970000E 04          1           0.00136
616              0.15040000E 04          1           0.00136
617              0.15170000E 04          1           0.00136
618              0.15340000E 04          1           0.00136
619              0.15350000E 04          2           0.00272
621              0.15620000E 04          1           0.00136
622              0.15940000E 04          1           0.00136
623              0.15950000E 04          1           0.00136
624              0.16030000E 04          1           0.00136
625              0.16520000E 04          1           0.00136
626              0.16530000E 04          1           0.00136
627              0.16930000E 04          1           0.00136
628              0.17000000E 04          1           0.00136
629              0.17150000E 04          1           0.00136
630              0.17500000E 04          1           0.00136
631              0.17550000E 04          1           0.00136
632              0.17860000E 04          1           0.00136
633              0.18220000E 04          1           0.00136
634              0.18240000E 04          1           0.00136
635              0.18550000E 04          1           0.00136
636              0.18770000E 04          1           0.00136
637              0.18930000E 04          1           0.00136
638              0.18970000E 04          1           0.00136
639              0.19230000E 04          1           0.00136
640              0.19640000E 04          1           0.00136
641              0.19650000E 04          1           0.00136
642              0.20510000E 04          1           0.00136
643              0.20630000E 04          1           0.00136
644              0.21670000E 04          1           0.00136
645              0.21850000E 04          2           0.00272
647              0.22080000E 04          1           0.00136
648              0.22530000E 04          1           0.00136
649              0.22600000E 04          1           0.00136
650              0.22710000E 04          1           0.00136
651              0.22870000E 04          1           0.00136
652              0.23160000E 04          1           0.00136
653              0.23210000E 04          1           0.00136
654              0.24290000E 04          1           0.00136
655              0.24390000E 04          1           0.00136
```

Figure 20c - USING LIST WITH TELEPHONE DATA SET 2
(cont.)

```
656              0.24680000E 04          1          0.00136
657              0.24720000E 04          1          0.00136
658              0.24900000E 04          1          0.00136
659              0.26840000E 04          1          0.00136
660              0.27930000E 04          1          0.00136
661              0.28820000E 04          1          0.00136
662              0.28910000E 04          1          0.00136
663              0.29380000E 04          1          0.00136
664              0.30800000E 04          1          0.00136
665              0.31170000E 04          1          0.00136
666              0.31650000E 04          1          0.00136
667              0.32860000E 04          1          0.00136
668              0.33150000E 04          1          0.00136
669              0.33780000E 04          1          0.00136
670              0.35100000E 04          1          0.00136
671              0.38070000E 04          1          0.00136
672              0.38370000E 04          1          0.00136
673              0.38500000E 04          1          0.00136
674              0.38580000E 04          1          0.00136
675              0.38790000E 04          1          0.00136
676              0.38880000E 04          1          0.00136
677              0.39440000E 04          1          0.00136
678              0.39780000E 04          1          0.00136
679              0.43730000E 04          1          0.00136
680              0.43930000E 04          1          0.00136
681              0.45120000E 04          1          0.00136
682              0.46370000E 04          1          0.00136
683              0.47590000E 04          1          0.00136
684              0.48350000E 04          1          0.00136
685              0.52160000E 04          1          0.00136
686              0.54920000E 04          1          0.00136
687              0.56520000E 04          1          0.00136
688              0.57540000E 04          1          0.00136
689              0.61850000E 04          1          0.00136
690              0.62410000E 04          1          0.00136
691              0.63720000E 04          1          0.00136
692              0.63850000E 04          1          0.00136
693              0.66590000E 04          1          0.00136
694              0.68160000E 04          1          0.00136
695              0.68210000E 04          1          0.00136
696              0.73070000E 04          1          0.00136
697              0.73290000E 04          1          0.00136
698              0.77460000E 04          1          0.00136
699              0.79270000E 04          1          0.00136
700              0.80390000E 04          1          0.00136
701              0.80580000E 04          1          0.00136
702              0.83530000E 04          1          0.00136
703              0.88470000E 04          1          0.00136
704              0.92060000E 04          1          0.00136
705              0.92560000E 04          1          0.00136
706              0.95170000E 04          1          0.00136
707              0.95410000E 04          1          0.00136
708              0.95620000E 04          1          0.00136
709              0.99650000E 04          1          0.00136
710              0.10020000E 05          1          0.00136
711              0.10376000E 05          1          0.00136
712              0.10837000E 05          1          0.00136
713              0.10918000E 05          1          0.00136
714              0.12042000E 05          1          0.00136
715              0.12793000E 05          1          0.00136
716              0.13012000E 05          1          0.00136
717              0.13179000E 05          1          0.00136
718              0.14692000E 05          1          0.00136
719              0.15825000E 05          1          0.00136
720              0.16877000E 05          1          0.00136
721              0.18885000E 05          1          0.00136
722              0.19273000E 05          1          0.00136
723              0.19696000E 05          1          0.00136
724              0.19849000E 05          1          0.00136
725              0.19851000E 05          1          0.00136
726              0.20846000E 05          1          0.00136
727              0.21440000E 05          1          0.00136
728              0.24573000E 05          1          0.00136
729              0.24770000E 05          1          0.00136
730              0.26278000E 05          1          0.00136
731              0.27238000E 05          1          0.00136
732              0.28133000E 05          1          0.00136
733              0.29913000E 05          1          0.00136
734              0.39867000E 05          1          0.00136
735              0.53138000E 05          1          0.00136
736              0.67271000E 05          1          0.00136
```

Figure 20d - USING LIST WITH TELEPHONE DATA SET 2 (cont.)

V. ASSESSMENT OF VARIABILITY ROUTINES

A. INTRODUCTION

In the previous sections Probability Plotting Subroutines were presented for use in informal estimation of the form of the distribution of a set of independent observations. In the present section subroutines computing the variability of some basic statistics are described.

We know that $\bar{X}$ (sample mean) is an unbiased estimator of the population mean ($= x f_X(x)\,dx$), where X is a population with unknown distribution ($F_X(x)$) and unknown mean, and $X_1, X_2, \ldots, X_n$ is a sample of independent observations from X. Furthermore we can compute the Variance of $\bar{X}$ as:

$$\mathrm{Var}[\bar{X}] = \mathrm{Var}[\sum_{i=1}^{n} \frac{X_i}{n}]$$

$$= \frac{1}{n^2} \mathrm{Var}[\sum_{i=1}^{n} X_i]$$

$$= \frac{n}{n^2} \mathrm{Var}[X] = \frac{\sigma^2}{n} .$$

Thus we see it is possible to estimate the Var[$\bar{X}$], even if the population Variance (σ^2) is unknown, by simply using the sample variance S^2 as follows: Var[$\bar{X}$] = S^2/n, where $S^2 = \sum_{i=1}^{n} (X_i - \bar{X})^2 / (n-1)$ (an unbiased estimator of σ^2). Therefore S^2/n is an unbiased estimator of Var[$\bar{X}$] also.

Furthermore, especially for a Normal population, we can obtain a confidence interval for the mean using the t-statistic.

But, although for $\bar{X}$ there exists a direct assessment of variability, for estimates of other population parameters such as Skewness, Kurtosis, Coefficient of Variation, and so forth, this is not so simply obtained. Thus several methods have been introduced to obtain assessments of variability and for two of them (Sectioning of data and the Jacknife) the Subroutines 'SECTN' and 'JACK' will be presented.

B. SUBROUTINE SECTN

1. Description

'SECTN' Subroutine is used for assessing variability of estimates from data based on the Sectioning Method.

The basic idea of this method is:

Assume we have n independent observations X_1, X_2, ..., X_n from a population with unknown distribution function $F_X(x)$. Let θ be a parameter of $F_X(x)$ and $\theta(n) = \theta(X_1, X_2, \ldots, X_n)$ be a statistic which estimates θ.

Now we want to estimate the variance of $\tilde{\vartheta}(n)$ which is a new Random Variable and we work as follows:

1. Divide the sample into r disjoint sections of size m (r should be such that n=mr. If this is not possible some of the latter data is discarded).
2. For each section form the same estimate $\tilde{\vartheta}_i(m)$, $i = 1,2,\dots,r$
3. Compute the average of $\tilde{\vartheta}_i(m)$, that is:

$$\bar{\vartheta}(n) = \frac{1}{r} \sum_{i=1}^{r} \tilde{\vartheta}_i(m) \quad ,$$

doing so, we have an estimate of ϑ and

$$E[\bar{\vartheta}(n)] = E[\frac{1}{r} \sum_{i=1}^{r} \tilde{\vartheta}_i(m)]$$

$$= \frac{1}{r} rE[\tilde{\vartheta}(m)]$$

$$= E[\tilde{\vartheta}(m)] \quad ,$$

also we have:

$$Var[\bar{\vartheta}(n)] = Var[\frac{1}{r} \sum_{i=1}^{r} \tilde{\vartheta}_i(m)]$$

$$= \frac{1}{r^2} Var[\sum_{i=1}^{r} \tilde{\vartheta}_i(m)]$$

$$= \frac{1}{r} Var[\tilde{\vartheta}(m)]$$

$$= \frac{1}{r} \frac{1}{r-1} \sum_{i=1}^{r} (\tilde{\vartheta}_i(m) - \bar{\vartheta}(n))^2$$

$$= \frac{1}{r} S^2_{\tilde{\vartheta}(m)} \quad .$$

Thus we see that Var[$\tilde{\vartheta}(m)$] and Var[$\bar{\vartheta}(n)$] can be unbiasedly estimated from the sample variance of $\tilde{\vartheta}_i(m)$, the main advantage and the purpose of the sectioning of data method.

But the main disadvantage of this method is that we would like r to be as large as possible in order to make the variability of the variance estimate S^2 as small as possible. This however may be worse for the bias properties of the estimation procedure. Therefore the choice of r is a factor that should be considered.

It should be noted here that if (m) are approximately normal variates (This can be tested by the previous described Subroutine 'NORMPL') then confidence intervals for the unknown parameter can be obtained based on the t-Statistic, in the following way:

$$\bar{\vartheta}(n) \pm \frac{S_{\bar{\vartheta}(n)}}{r^{1/2}}\, t_{(1-\alpha/2),(r-1)}$$

where:

$\bar{\vartheta}(n)$ is the mean of the sectioned data statistics obtained from the column named 'mean' of the second table ('estimated parameters of the sample parameters') of the program output.

$S_{\bar{\vartheta}(n)}/r^{1/2}$ is the standard deviation of the sectioned data statistics divided by the square root of the number of sections, obtained from the last column of the same table of the program output.

$t_{(1-\alpha/2),(r-1)}$ is the $(1-\alpha/2)$ quantile of the t-distribution with r-1 degrees of freedom.

2. Program Structure

'SECTN', using the Sectioning method, estimates the following statistics: Mean, Median, Variance, Standard Deviation, Coefficient of Variation, Skewness, Kurtosis, Minimum, and Maximum. (The formulas which have been used to compute these estimates are described in the comments of the Subroutine.) The first table is then printed by the program, containing the values of these parameters. Then 'SECTN', using the computed estimates of all sections for each parameter, estimates the Mean, Median, Variance, Skewness, Kurtosis, and Standard Deviation divided by the square root of the number of sections. The second table is printed containing these values.

There are three restrictions in using 'SECTN':

1. The number of sections must be no greater than 100. If it is, a diagnostic message is printed and only estimates from unsectioned data will be given.
2. The number of data values must be greater than 3; otherwise a diagnostic message is printed without any calculation.
3. The size of each section must be greater than 3. If it is less than, or equal to 3, then the program gives estimates for the entire set of (unsectioned) data. A diagnostic is printed.

A complete description of how 'SECTN' operates is given in the Subroutine. Furthermore, a Summary is given by typing on the terminal the command DESCRIBE SECTN under the CMS environment. The following response will be given on the terminal when the user types the above command:

SUBROUTINE SECTN

'SECTN' is intended to estimate a set of basic statistics of a given set of observations using the 'Sectioning of data method'. Also for each estimated statistic, estimates of some basic statistics such as the mean, standard Deviation, and so forth, are given.

It is called by :

```
CALL   SECTN  ( X, N, K )
```

where:

X   Is the array of data

N   Is the number of data values (must be greater than 3)

K   Is the number of desired sections (no greater than 100)

Note: k should be a number which minimizes the number of data points that will have to be discarded. 'SECTN' places the data into the equal size sections discarding any data left over.

For $k \leq 3$ or $k > 100$ or $(n/k) \leq 3$ only estimates from unsectioned data will be given and no estimates for the estimated statistics will be computed.

No output is expected if $n \leq 3$.

More information is given in the subroutine.

3. Using 'SECIN' with Telephone Data Set 1

'SECIN' was used on Telephone Data Set 1 to assess the variability in the Mean, Median, Variance, Standard Deviation, Coefficient of Variation, Skewness and Kurtosis.

The 672 data-points of Telephone Data Set 1 were broken down into 16 disjoint sections with 42 data-points per section. Because of this break-down no data-points were discarded. Compare the values of row 'unsectioned' (see figure 21) with the values of the corresponding statistics computed by the Subroutine 'HISTGS/HISTFS' (see figure 1). The values are the same.

Now if we want to assess the variability in any of the above parameters we proceed as follows:

1. We take the mean (ϑ) of the parameter, whose the variability we want to assess, from the table under 'ESTIMATED PARAMETERS OF SAMPLE PARAMETERS'.
2. We take, from the same table, the value (S) from the last column (STD. DEV./NS**.5) for the same parameter.
3. Using the t-statistic with k-1 d.f. and the formula:

$$\vartheta \pm St_{(1-\alpha/2),(k-1)}$$

we get a $(1-\alpha)$% confidence interval for the parameter.

Example

In order to find a 95% confidence interval for the Skewness of Telephone Data 1 we take:

$\vartheta = 4.9343$, $S = .30312$, $t_{.975,15} = 2.131$.

Therefore the 95% confidence interval is:

$$4.9343 \pm .30312*2.131 = 4.9343 \pm .64595$$

$$\Longrightarrow [4.2884,\ 5.5802].$$

It should be noted here that the use of the Variance estimate from the sectioned data in order to get confidence intervals of the parameters is based on the normality assumption and the independence of the estimates from the sections. The normality will depend on the number of data-points in each section, which should be kept large. This requirement, however, conflicts with the need to make the number of sections large to reduce the variability in the estimate of the variance of the statistics. The skewness estimates of each of the 16 sections, can be run through the normal plotting routine to see whether the use of the t-statistic confidence interval is valid.

```
                                          ESTIMATED SAMPLE PARAMETERS

  SECTION      MEAN        MEDIAN      VARIANCE    STD. DEV.    COEF VAR    SKEWNESS    KURTOSSIS    MINIMUM      MAXIMUM

      1     1.0526E 03  8.5000E 00  3.4598E 07  5.8820E 03  5.5879E 00  6.3484E 00  3.7831E 01  1.0000E 00  3.8003E 04
      2     3.2135E 03  1.4500E 01  1.8494E 08  1.3599E 04  4.2320E 00  5.8106E 00  3.3017E 01  1.0000E 00  8.5993E 04
      3     1.7662E 03  1.4500E 01  4.2383E 07  6.5103E 03  3.6860E 00  4.2806E 00  1.7836E 01  1.0000E 00  3.5644E 04
      4     6.0669E 02  1.1000E 01  5.3412E 06  2.3111E 03  3.8094E 00  4.3148E 00  1.6486E 01  1.0000E 00  1.1280E 04
      5     1.5639E 03  5.0500E 01  2.1924E 07  4.6824E 03  2.9941E 00  4.2209E 00  1.8565E 01  1.0000E 00  2.6443E 04
      6     2.5343E 03  5.7000E 01  4.0337E 07  6.3511E 03  2.5061E 00  3.1573E 00  9.5654E 00  1.0000E 00  3.0974E 04
      7     2.6778E 03  2.2000E 01  7.2756E 07  8.5297E 03  3.1853E 00  4.1587E 00  1.7579E 01  1.0000E 00  4.7120E 04
      8     9.8881E 02  1.8500E 01  3.8282E 07  6.1873E 03  6.2573E 00  6.4801E 00  3.8995E 01  1.0000E 00  4.0131E 04
      9     1.5176E 03  2.2000E 01  2.0792E 07  4.5599E 03  3.0046E 00  2.9866E 00  6.8551E 00  1.0000E 00  1.7174E 04
     10     2.7682E 03  1.4000E 01  1.0906E 08  1.0443E 04  3.7726E 00  4.8932E 00  2.4134E 01  1.0000E 00  6.1710E 04
     11     1.9258E 03  1.4000E 01  5.9852E 07  7.7364E 03  4.0173E 00  5.4134E 00  2.9059E 01  1.0000E 00  4.7592E 04
     12     8.1955E 02  4.9500E 01  8.2895E 06  2.8791E 03  3.5131E 00  4.5999E 00  1.9785E 01  1.0000E 00  1.5868E 04
     13     2.1201E 03  4.0000E 00  1.2224E 08  1.1056E 04  5.2150E 00  5.9400E 00  3.3861E 01  1.0000E 00  6.9775E 04
     14     2.3062E 02  1.1500E 01  3.3035E 05  5.7476E 02  2.4923E 00  3.4695E 00  1.2010E 01  1.0000E 00  2.9620E 03
     15     4.3752E 02  7.0000E 00  5.7201E 06  2.3917E 03  5.4664E 00  6.3983E 00  3.8289E 01  1.0000E 00  1.5504E 04
     16     5.4838E 02  6.5000E 00  1.1340E 07  3.3675E 03  6.1408E 00  6.4765E 00  3.8964E 01  1.0000E 00  2.1848E 04

UNSECTIONED 1.5482E 03  1.4000E 01  4.8362E 07  6.9543E 03  4.4918E 00  7.1531E 00  6.2609E 01  1.0000E 00  8.5993E 04


                                   ESTIMATED PARAMETERS OF SAMPLE PARAMETERS

PARAMETER       MEAN         MEDIAN       VARIANCE     STD. DEV.     COEF VAR      SKEWNESS     KURTOSSIS  STD. DEV./NS**.5

MEAN         1.5482E 03   1.5408E 03   8.6488E 05   9.2999E 02   6.0069E-01    2.7558E-01  -1.2951E 00     2.3250E 02
MEDIAN       2.0313E 01   1.4250E 01   2.8023E 02   1.6740E 01   8.2412E-01    1.4356E 00   3.0546E-01     4.1850E 00
VARIANCE     4.8637E 07   3.6440E 07   2.6217E 15   5.1203E 07   1.0528E 00    1.5503E 00   1.4675E 00     1.2801E 07
STD. DEV.    6.0664E 03   6.0346E 03   1.2625E 07   3.5532E 03   5.8572E-01    5.5874E-01  -4.7073E-01     8.8830E 02
COEF. VAR.   4.1175E 00   3.7910E 00   1.5503E 00   1.2451E 00   3.0240E-01    4.9658E-01  -1.2011E 00     3.1128E-01
SKEWNESS     4.9343E 00   4.7465E 00   1.4701E 00   1.2125E 00   2.4573E-01   -1.0259E-01  -1.4647E 00     3.0312E-01
KURTOSSIS    2.4552E 01   2.1960E 01   1.2484E 02   1.1173E 01   4.5508E-01    4.0534E-03  -1.5801E 00     2.7933E 00

                                                                                                        NS=# OF SECTS
```

Figure 21 - USING SECTN WITH TELEPHONE DATA SET 1.

C. SUBROUTINE JACK

1. Description

This routine is used for assessing variability of estimates of data based on the Jacknife method, and for reducing the bias in estimates also. It is particularly useful for data with small sample size.

A big picture of the method has as follows:

Let $X_1, X_2, \ldots, X_n$ be a sample of n independent, identically distributed observations from a population with unknown distribution function $F_X(x)$. Also let ϑ be an unknown parameter of $F_X(x)$ to be estimated. Furthermore suppose a method (biased or unbiased) is available for estimating ϑ. Then we proceed as follows:

1. Divide the sample size into r disjoint groups, each of size m (r should be such that n = mr. If this is not possible some of the last data-points will be discarded.)
2. Compute the estimator $\tilde{\vartheta}(n)$ of ϑ based on all n = mr observations.
3. Compute the estimator $\tilde{\vartheta}_i(n-m)$ based on the n-m observations, having deleted the i^{th} group.
4. Compute the so called 'PSEUDO VALUES' $\hat{\vartheta}_i(n)$:

$$\hat{\vartheta}_i(n) = r\,\tilde{\vartheta}(n) - (r-1)\,\tilde{\vartheta}_i(n-m) , \quad i = 1,2,\ldots,r .$$

Then the Jacknife estimator is defined to be the average of Pseudo Values:

$$\hat{\vartheta}(n) = \frac{1}{r} \sum_{i=1}^{r} \hat{\vartheta}_i(n)$$

$$= \frac{1}{r} \left(r^2 \tilde{\vartheta}(n) - (r-1) \sum_{i=1}^{r} \tilde{\vartheta}_i(n-m) \right)$$

$$= r \tilde{\vartheta}(n) - \frac{r-1}{r} \sum_{i=1}^{r} \tilde{\vartheta}_i(n-m) \quad .$$

Having so defined the Jacknife estimator it can be proved that this estimator is an unbiased estimator of ϑ, except for terms n^{-2} and higher order, assuming that the bias has the form of $E[\tilde{\vartheta}(n)] = \vartheta + a/n +$ terms of higher order in n. That is, the Jacknife estimator eliminates a n^{-1} bias term. Namely if

$$E[\tilde{\vartheta}(n)] = \vartheta + an^{-1} + O(n^{-2}) \quad ,$$

then we have:

$$E[\hat{\vartheta}(n)] = E[r\tilde{\vartheta}(n) - \frac{r-1}{r} \sum_{i=1}^{r} \tilde{\vartheta}_i(n-m)]$$

$$= rE[\tilde{\vartheta}(n)] - \frac{r-1}{r} \sum_{i=1}^{r} E[\tilde{\vartheta}_i(n-m)]$$

$$= r(\vartheta + an^{-1} + O(n^{-2})) - \frac{r-1}{r}(r\vartheta + \frac{ra}{n-m} + \frac{r}{n-m} O(n^{-2}))$$

$$= r\vartheta + \frac{ra}{n} + \ldots - (r-1)(\vartheta + \frac{a}{m(r-1)} + \ldots)$$

$$= r\vartheta + \frac{ra}{n} + \ldots - r\vartheta - \frac{(r-1)a}{m(r-1)} + \vartheta + \ldots$$

$$= r\vartheta - r\vartheta + \frac{ra}{mr} - \frac{a}{m} + \vartheta + \ldots$$

$$= \vartheta .$$

A special case of Jacknife estimator can be mentioned here, called the 'Complete Jacknife Estimator', where $r=n$ and $m=1$.

Some properties of the Jacknife estimator follow:

1. It is unbiased up to order n^{-2}.
2. The 'Pseudo values' can be used to obtain variance estimates for the Jacknife estimator since they can be considered (see [18]) as approximately independent and identically distributed. Thus $(r(r-1))^{-1} \sum_{i=1}^{r} (\hat{\vartheta}_i(n) - \hat{\vartheta}(n))^2$ should be an approximate estimate of $\mathrm{Var}[\hat{\vartheta}(r)]$,

and

$$\frac{(\hat{\vartheta}(n) - \vartheta)\ (r(r-1))^{1/2}}{\{\sum_{i=1}^{r} (\hat{\vartheta}_i(n) - \hat{\vartheta}(n))^2\}^{1/2}}$$

should be approximatelly distributed as a t-statistic with $r-1$ d.f. This procedure is particularly useful if the number n of data-points is small, but it must be used with care.

3. For large n it can be shown that for the complete Jacknife estimator, under very general conditions we have:

$$\mathrm{Var}[\hat{\vartheta}(n)] \ \text{--->}\ \mathrm{Var}[\tilde{\vartheta}(n)] .$$

2. Program Structure

Subroutine 'JACK' is a FORTRAN-callable Subroutine which takes a set of data, groups it into r disjoint groups and for each set of the r-ith group computes and prints in a table the statistics: Mean, Median, Variance, Standard Deviation, Coefficient of Variation, Skewness and Kurtosis. The same statistics are also computed for the ungrouped set of data. Then, using these statistics, the Jacknife estimator of the above parameters are computed and printed in another table along with its Variance and Standard Deviation.

The program is divided into the main program and the Subroutine JACKES. The main program groups the data and successively calls the Subroutine 'JACKES' in order to get the estimates of the above parameters for ungrouped data and for each group as well. A table is then printed containing the estimated parameters. The 'Pseudo-Values' for each parameter are also computed by the main program, and are used to compute the Jacknife estimator for each parameter and its Variance and Standard Deviation.

To avoid division by zero the number of observations must be greater than three, otherwise no output is given. For the same reason the expression (r-1)(n/r) must evaluate to greater than three, otherwise the program will give only estimates of ungrouped data.

A complete description of how 'JACK' operates is given in the Subroutine. A Summary is printed on the terminal by typing DESCRIBE JACK under the CMS environment, which responds:

SUBROUTINE JACK

Jack is intended to estimate the statistics : Mean, Median, Variance, Standard Deviation, Coefficient of Variation, Skewness and Kurtosis of a given set of independent observations, using the Jacknife method. In addition to the above parameters which are given for each group, the Jacknife estimator along with its variance and standard deviation is given for each parameter.

It is called by:

```
CALL   JACK  ( X, XS, STAT, N, IG )
```

where:

X	is the array of data of dimension N
XS	is a work array of dimension N (returns ordered data)
STAT	is a two-dimensional work array, dimensioned by (IG,7)
N	is the number of data-values
IG	is the number of groups plus one (=r+1)

Note : r must be such a number as to minimize the number of data-points that will have to be discarded. JACK places the data into the equal size groups discarding any data left over (last observations in order of original input).

No output is expected if $N \le 3$. Furthermore if $IG \le 2$ or $(IG-2)(N/(IG-1)) \le 3$ only estimates for ungrouped data will be given.

More information is given in the Subroutine.

3. Using 'JACK' with Telephone Data 1

To assess the variability in the Mean, Median, Variance, Standard Deviation, Coefficient of Variation, Skewness and Kurtosis of Telepfone Data 1 the Subroutine 'JACK' is now used. In order to avoid to discarding any of the data-points we use 16 groups with 42 data-points per group.

Observing figure 22 we see that the values of the parameters for ungrouped data are exactly the same as the corresponding values of the same parameters of figure 1 which was produced by using the Subroutine HISTGS/HISTFS. Now in order to assess the variability in any of the above parameters we have to use the t-statistic with (r-1) d.f. along with the values printed under 'ESTIMATED JACKNIFE PARAMETERS' using:

$$\vartheta \pm S t_{(1-\alpha/2),(r-1)}$$

where ϑ is the Jacknife estimator and S is its Standard Deviation. For example to assess the variability of the Skewness with a confidence level $\alpha = .05$ we have:

$$\vartheta = 7.37321, \qquad S = .900118, \qquad t_{.975,15} = 2.131 ,$$

Therefore a 95% confidence interval of the Skewness is:

$$7.37321 \pm .900118*2.131 = 7.37321 \pm 1.91815$$

$$\Longrightarrow [5.4551, 9.2914] .$$

This should be compared with the point estimate 4.9343 and the confidence interval estimate [4.2884, 5.5802] obtained from subroutine SECTN. The data is so skewed that one will inevitably have trouble here with the Jacknife procedure.

```
                                          ESTIMATED PARAMETERS

   GROUP        MEAN         MEDIAN        VARIANCE      STD. DEV.     COEF. VAR     SKEWNESS      KURTOSSIS

     1      1.58126E 03   1.50000E 01   4.93184E 07   7.02270E 03   4.44121E 00   7.17459E 00   6.30249E 01
     2      1.43720E 03   1.40000E 01   3.93384E 07   6.27203E 03   4.36406E 00   6.52195E 00   5.06898E 01
     3      1.53369E 03   1.40000E 01   4.88250E 07   6.98749E 03   4.55601E 00   7.30927E 00   6.47616E 01
     4      1.61099E 03   1.40000E 01   5.11798E 07   7.15400E 03   4.44076E 00   6.97807E 00   5.92573E 01
     5      1.54717E 03   1.35000E 01   5.01620E 07   7.08251E 03   4.57771E 00   7.14944E 00   6.18267E 01
     6      1.48248E 03   1.30000E 01   4.88925E 07   6.99232E 03   4.71663E 00   7.36629E 00   6.51602E 01
     7      1.47291E 03   1.30000E 01   4.67577E 07   6.83796E 03   4.64248E 00   7.50811E 00   6.88309E 01
     8      1.58551E 03   1.40000E 01   4.90734E 07   7.00524E 03   4.41828E 00   7.18504E 00   6.33708E 01
     9      1.55026E 03   1.35000E 01   5.02357E 07   7.08771E 03   4.57196E 00   7.15921E 00   6.17726E 01
    10      1.46689E 03   1.40000E 01   4.43761E 07   6.66154E 03   4.54127E 00   7.45724E 00   6.96176E 01
    11      1.52305E 03   1.40000E 01   4.76796E 07   6.90504E 03   4.53370E 00   7.32003E 00   6.59486E 01
    12      1.59680E 03   1.30000E 01   5.10129E 07   7.14233E 03   4.47291E 00   7.00921E 00   5.97006E 01
    13      1.51010E 03   1.60000E 01   4.35998E 07   6.60302E 03   4.37258E 00   7.14932E 00   6.50323E 01
    14      1.63606E 03   1.40000E 01   5.14459E 07   7.17258E 03   4.38406E 00   6.92004E 00   5.85264E 01
    15      1.62226E 03   1.50000E 01   5.11303E 07   7.15055E 03   4.40775E 00   6.97844E 00   5.93194E 01
    16      1.61487E 03   1.50000E 01   5.07807E 07   7.12606E 03   4.41276E 00   7.02911E 00   6.01293E 01

 UNGROUPED  1.54822E 03   1.40000E 01   4.83618E 07   6.95427E 03   4.49178E 00   7.15313E 00   6.26085E 01


              ESTIMATED JACKKNIFE PARAMETERS

 PARAMETER      JACKKNIFE      VARIANCE       STD. DEV. OF
                                            JACK. PARAMETER

 MEAN          1.54825E 03   8.64880E 05    2.32497E 02
 MEDIAN        1.30625E 01   1.56563E 02    3.12812E 00
 VARIANCE      4.83451E 07   2.54534E 15    1.26128E 07
 STD. DEV.     7.01556E 03   1.39786E 07    9.31350E 02
 COEF. VAR.    4.50542E 00   2.42612E 00    3.89400E-01
 SKEWNESS      7.37321E 00   1.29634E 01    9.00118E-01
 KURTOSSIS     6.70793E 01   4.68056E 03    1.71037E 01
```

Figure 22 - USING JACK WITH TELEPHONE DATA SET 1.

VI. SINGLE-SERVER FIRST-COME FIRST-SERVED QUEUES WITH 'EARMA' STRUCTURE

A. THE EARMA(P,Q) PROCESS

The starting point for these processes is the definition of a first-order autoregressive model for a stationary sequence of random variables $\{X_i\}$:

$$X_i = rX_{i-1} + \varepsilon_i ; \qquad i = 0, \pm 1, \pm 2, \ldots .$$

If the marginal distribution of the X_i is fixed to be exponential with parameter λ for all i;

$$P\{X_i \leq x\} = 1 - e^{-\lambda x}, \qquad \lambda > 0,\ x \geq 0 ,$$

then ε_i is zero with probability r and exponential(λ) with probability 1-r. Thus

$$X_i = \begin{cases} rX_{i-1} & \text{w.p. } r \\ rX_{i-1} + E_i & \text{w.p. } 1-r, \end{cases} \qquad (1)$$

where $\{E_i\}$ is a sequence of i.i.d. exponential(λ) random variables. The process defined by (1) is the exponential autoregressive process of order 1, the EAR(1) process. The correlation structure is $r(j) = r^j$, $j = 0,1,2,\ldots$.

The first order moving average exponential process, EMA(1) is defined as

$$X_i = \begin{cases} \beta E_i & \text{w.p. } \beta \\ \beta E_i + E_{i-1} & \text{w.p. } (1-\beta) \end{cases} \qquad (0 \le \beta \le 1; \quad i = 0, \pm 1, \pm 2, \ldots)$$

and has correlations $\rho(1) = (1-\beta)$ and $\rho(1) = 0$, $j = 2,3,\ldots$

The general EMA(q) model takes the form

$$X_i = \begin{cases} \beta_q E_i & \text{w.p. } b_{q+1} \\ \beta_q E_i + \beta_{q-1} E_{i-1} & \text{w.p. } b_q \\ \quad \ldots\ldots & \ldots\ldots \\ \beta_q E_i + \beta_{q-1} E_{i-1} + \ldots + \beta_1 E_{i-q+1} & \text{w.p. } b_2 \\ \beta_q E_i + \beta_{q-1} E_{i-1} + \ldots + \beta_1 E_{i-q+1} + E_{i-q} & \text{w.p. } b_1 \end{cases} \qquad (2)$$

for $0 \le \beta_1, \beta_2, \ldots, \beta_q \le 1$; $i = 0, \pm 2, \ldots$ where

$$b_i = \begin{cases} \beta_q & i = q+1, \\ (1-\beta_q) \ldots (1-\beta_i)\beta_{i-1} & q \ge i \ge 2, \\ (1-\beta_q) \ldots (1-\beta_i) & i = 1 \quad . \end{cases}$$

The serial correlations are given by the equation

$$\rho^{(q)}(j) = \mathrm{corr}(X_i, X_{i-r}) = \begin{cases} \sum_{v=j}^{q-j+1} b_v b_{v+j} & (1 \le j \le q) \\ 0 & (q+1 \le j < \infty) \end{cases}$$

To convert this EMA(q) process into a mixed autoregressive moving-average exponential process of orders 1 and q, called EMA(1,q) we replace E_{i-q} in (2) with

$$A_{i-q} = \begin{cases} rA_{i-q-1} & \text{w.p. } r \\ rA_{i-q-1} + E_{i-q} & \text{w.p. } (1-r) \end{cases}$$

For further results on these processes and their properties see Gaver and Lewis (1978), Lawrance and Lewis (1977), Jacobs and Lewis (1977), and Lawrance and Lewis (1978).

E. USE OF AUTOREGRESSIVE (EAR(P)), MOVING AVERAGE (EMA(Q)) AND MIXED STRUCTURES (EARMA(P,Q)) IN MODELLING QUEUES

Consider for simplicity a queue with a single input stream and a single server, and a first-come-first-served (FIFO) service discipline. Let S_i, $i = 0,1,2,\ldots$, denote the service time for the i^{th} arrival, and let X_i, $i = 1,2,\ldots$, denote times between arrival of the i^{th} and $(i-1)^{st}$ customers. As is usual we assume that the first customer (with service time S_0) arrives at time zero and finds the queue empty.

If the $\{S_i\}$ and $\{X_i\}$ sequences are i.i.d. exponential random variables with parameters λ and α respectively, we have the M/M/1 queue.

Now let

E_i be exponential(λ) and independent, $i = 0,+1,\ldots$;

$\mathcal{E}_i$ be exponential(α) and independent, $i = 0,+1,\ldots$.

We want to model queues with correlated (autocorrelated and/or cross-correlated) service and inter-arrival times, the service and inter-arrival times both having marginally exponential distributions. We also want the queing model to include the M/M/1 queue as a special case.

There are five simple possibilities which we put forward here, based on the use of EARMA(p,q) processes, giving what we call an EARMA MD/MD/I queue.

In what follows let p_S and p_X be, respectively, the order of the autoregressive components of the service time sequence $\{S_i\}$ and the inter-arrival time sequence $\{X_i\}$, and let q_S,q_X be, respectively, the order of the moving average component of $\{S_i\}$ and $\{X_i\}$. These parameters can take values $0,1,\ldots$. If $p = 0$, $q = 0$, the sequence is independent; if $p = 0$, the process is purely moving average and if $q = 0$, then the process is purely autcregresive.

The five possibilities or cases are as follows:

1) Let $\{S_i\}$ be EARMA(p_S,q_S) over $\{E_i, E_{i-1}, \ldots\}$;

let $\{X_i\} = \{\mathcal{E}_i\}$.

Thus the arrivals are a Poisson process and the service times are autocorrelated, i.e. a mixed autoregressive moving average exponential sequence.

2) Let $S_i = E_i$, $i = 0,1,2, \ldots$;

let X_i be EARMA(p_X, q_X) over $\{\mathcal{E}_i, \mathcal{E}_{i-1}, \ldots\}$.

Then the service times are independent and the arrival process is non-Poisson because of the dependency between the inter-arrival times, i.e. the arrival process is a point process with EARMA(p,q) structures.

3) Let $\{S_i\}$ be EARMA(p_S, q_S) over $\{E_i, E_{i-1}, \ldots\}$.

Let $\{X_i\}$ be EARMA(p_X, q_X) over $\{\mathcal{E}_i, \mathcal{E}_{i-1}, \ldots\}$.

The service and arrival processes are autocorrelated, but the two processes are independent. The marginal distributions of $\{S_i\}$ and $\{X_i\}$ are still exponential.

4) To couple the two processes, with resultant dependence in the arrival and service processes, the simplest procedure seems to be the following.

Let $\{S_i\}$ be EARMA(p_S, q_S) in the following sense:

$\{S_i\}$ is EARMA(p_S, q_S) over $\{E_i, \frac{\alpha}{\lambda}\mathcal{E}_i, \frac{\alpha}{\lambda}\mathcal{E}_{i-1}, \ldots\}$.

Then if $X_i = \mathcal{E}_i$, $i = 0,1,2,\ldots$ we have that $\{S_i\}$ is an autocorrelated sequence and also cross-correlated with $\{X_i = \mathcal{E}_i\}$, i.e. $\{S_i, X_i\}$ is a bivariate dependent sequence of random variables with exponential marginal distributions, although $\{X_i\}$ is still Poisson. Note that the S_i sequence is an autocorrelated sequence because of the cross-coupling, but not a pure

EARMA(p_S, q_S) sequence. It does have an exponential marginal distribution however.

As an example let $p_S = 1$, $q_S = 1$.

Let

$$S_0 = E_0$$

$$\begin{aligned}
S_1 &= \beta_S E_1 && \text{w.p. } \beta_S && A_1 = \tfrac{\alpha}{\lambda}\varepsilon_1 \\
&= \beta_S E_1 + A_1 && \text{w.p. } (1-\beta_S) && \\
S_2 &= \beta_S E_2 && \text{w.p. } \beta_S && A_2 = rA_1 && \text{w.p. } r \\
&= \beta_S E_2 + A_2 && \text{w.p. } (1-\beta_S) && = rA_1 + \tfrac{\alpha}{\lambda}\varepsilon_2 && \text{w.p. } (1-r)
\end{aligned}$$

.
.
.

$$\begin{aligned}
S_i &= \beta_S E_i && \text{w.p. } \beta_S && A_i = rA_{i-1} && \text{w.p. } r \\
&= \beta_S E_i + A_i && \text{w.p. } (1-{}_S) && = rA_{i-1} + \tfrac{\alpha}{\lambda}\varepsilon_i && \text{w.p. } (1-r)
\end{aligned}$$

.
.
.

In these equations one can replace ε_i with X_i and then the cross-coupling between the sequences $\{S_i\}$ and $\{X_i\}$ becomes apparent.

Interpretation. We have positive correlation between S_i and q_S previous inter-arrival times (when $p_S = 0$). If

the ε_j's ($j = i, i-1, \ldots, i-p_S-1$) are short, then S_i will be short if all the q_S previous inter-arrival times are short. *This models the case where the server tends to speed up if the queue gets long.* Of cource he also slows down when the queue gets short and it is not immediately clear what the effect on an average waiting time will be.

4') Similar to 4 but $\{X_i\}$ is EARMA(p_X, q_X) over $\varepsilon_i, \frac{\lambda}{\alpha}E_i, \frac{\lambda}{\alpha}E_{i-1}, \ldots$ and $S_i = E_i$.

Interpretation is not clear, but it would have the same effect as balking in the input stream. If service times get long (and presumably the queue gets long) then inter-arrival times get long.

5) A more general possibility which allows one to model dependency in the input stream, in the service stream and dependency which couples them is as follows:

Let $\{D_i\}$ be a unit exponential independent sequence, a driving sequence.

Let $\{\bar{E}_i\}$ be EARMA(p_S, q_S) over $\{E_i, E_{i-1}, \ldots\}$.

Let $\{\bar{X}_i\}$ be EARMA(p_X, q_X) over $\{\varepsilon_i, \varepsilon_{i-1}, \ldots\}$.

Let $\{S_i\}$ be EARMA(p_{C_1}, q_{C_1}) over $\{\bar{E}_i, \lambda D_i, \lambda D_{i-1}, \ldots\}$.

Let $\{X_i\}$ be EARMA(p_{C_2}, q_{C_2}) over $\{\bar{X}_i, \alpha D_i, \alpha D_{i-1}, \ldots\}$.

This is a very general scheme which reduces to all of the previous schemes as special cases. For example if $p_{C_1} = q_{C_1} = p_{C_2} = q_{C_2} = 0$, then we have case 3. If $q_{C_2} = 1$, $p_{C_2} = 0$, $\ell_{C_2} = 0$, $p_S = q_S = 0$; $p_X = 0$ (and p_{C_1} and q_{C_1} are not equal to zero) we have case 4.

In all cases the basic equation for M/M/1 queue waiting time W_n still holds:

$$W_0 = 0$$

$$W_{n+1} = (W_n + S_n - X_{n+1})^+ \qquad n = 1,2,\ldots .$$

These equations can be used to generate successive W_n's in a simulation. We have not touched on how the correlated sequences are started. This is somewhat arbitrary as it is not known (except for the EMA(1) and MA(1) processes) how to start them so as to produce a stationary exponential sequence. The problem is aggravated, as in cases 4, 4' and 5, when we are dealing with bivariate exponential sequences. Then the marginal processes may be stationary but not the bivariate process.

C. PROGRAM STRUCTURE

1. General

To simulate the EARMA(p,q) model, a FORTRAN Program has been written to take care of all the existing cases of that model and also of the M/M/1 queue. Depending on the input values which the parameters (as described below) will take, the program can be used for creating in each run a particular case of the EARMA(p,q) queue. Because of that generality of the program it is not suggested that it be used for those EARMA cases where either the arrival times or the service times or both are not autocorrelated or cross-correlated. The reason for this suggestion is efficiency. It is suggested for use with the EARMA(p,q) cases where both arrival and service times are autocorrelated and, furthermore, cross-correlated with any order. The program is divided into the main program and into the Subroutines BETAS, AUTOR and EARMA.

2. Main Program

The main program performs all I/O operations and calculations required for the desired statistics. Specifically it is designed:

a) To read

1. All random number generator seeds required to generate the sequences of the Exponential and Uniform variates to be used during the execution.

2. The values for the variables:

N : Number of arrivals to be generated.

M : Number of replications.
RX, RS : Arrival rate, service rate.
CPX, CPS, CPG : Taking the values 1 or 0 (1 for the coupled process).
KX1, KX, KS1, KS : The order of moving average for arrival and service processes.
QX1, QX, QS1, QS : Taking the values 1 or 0 (1 for autoregressive case).
BX, BX1, BS, BS1 : Arrays with the values of b's for moving average part.
RHOX1, RHOX, RHOS1, RHOS : Parameter values for the autoregressive parts.

b) To calculate and print

SUMX(I,J);	SUMXM(I,J);
SUMS(I,J);	SUMSM(I,J);
W(I,J);	WM(I,J);
WB(I,J);	WMB(I,J);
D(I,J);	DM(I,J);
for I = 1,2,3,4,	J = 1,2,...,M,

where:

(In what follows k gets the values N/4, N/2, 3N/4, N for I=1,2,3,4 respectively and J=1,2,...,M)

SUMX(I,J) = $\sum_{r=1}^{k} X_r^{(J)}$, $X_r^{(J)}$ the r^{th} arrival for the J^{th} replication in the EARMA queue;

SUMXM(I,J) = $\sum_{r=1}^{k} (XM)_r^{(J)}$, $(XM)_r^{(J)}$ the r^{th} arrival for the J^{th} replication in M/M/1 Queue;

SUMS(I,J) = $\sum_{r=1}^{k} S_r^{(J)}$, $S_r^{(J)}$ the r^{th} service for the J^{th} replication in the EARMA queue;

SUMSM(I,J) = $\sum_{r=1}^{k} (SM)_r^{(J)}$, $(SM)_r^{(J)}$ the r^{th} service for the J^{th} replication in M/M/1 Queue;

W(I,J) = $W_k^{(J)} = \max\{ (W_{k-1}^{(J)} + S_{k-1}^{(J)} - X_k^{(J)}), 0\}$

= waiting time of k^{th} arrival in J^{th} replication of the correlated queue;

WM(I,J) = $(WM)_k^{(J)}$ The same as W(I,J) but for M/M/1 queue;

WB(I,J) = $\sum_{r=1}^{k} W_r^{(J)} / k$

WMB(I,J) = $\sum_{r=1}^{k} (WM)_r^{(J)} / k$

D(I,J) = {SUMS(I,J) - SUMX(I,J)}/K - 1/RS + 1/RX

DM(I,J) = {SUMSM(I,J) - SUMXM(I,J)}/K - 1/RS +1/RX

SUMSM(I,J) = $\sum_{r=1}^{k} (SM)_r^{(J)}$,

Note: Between the group-variables (KX1, KX, KS1, KS), (QX1, QX, QS1, QS), (CPX, CPS, CPG), (RHOX1, RHOX, RHOS1, RHOS) and (BX1, BX, BS1, BS) there are the following relations:

1. The number of elements of each variable (array) from B-group should be the same as the value of the corresponding variable from K-group.

2. The '0' value of a variable from Q-group implies that the corresponding variable from R-group is not needed.

3. The value '1' of CPG dominates any value of CPX, or CPS.

Any valid combination in the values of the above variables specially of the Q-group, C-group and K-group (0 or not 0), generates a particular EARMA(p,q) case and is accepted by the program.

The basic approach in the main program is to generate at once and for each replication three independent exponential sequences with parameters XL (arrival rate), SI (service rate) and 1 (unit exponential) which are stored in the arrays EXPXL, EXPSI and EXP1 respectively. Then the Subroutine AUTOR which is involved with the autoregressive part of the model is called (if it is needed for the particular EARMA case). A loop follows which is executed as many times as the value of N. From the loop the EARMA Subroutine is called which is involved with the moving average part of the model. The number of calls of Subroutine EARMA depends on the particular case which is being run. The statistics and all information that will help us to analyze the model are also computed inside the loop and stored in the arrays described above. The program continues execution until the desired statistics for all M replications have been calculated and gathered. Then the output part of the program follows which gives us the values of calculated statistics on paper and on punched cards for analysis.

3. Subroutine BETAS

This is a simple Subroutine and its purpose is to return an array SUMBX1/X/S1/S with element-values as follows:

$$\mathrm{SUMBX}(I) = \sum_{r=1}^{I} \mathrm{EX}(r), \qquad I = 1,2,\ldots,\text{value of K-group}$$

to be used for the choice of the order in the moving average. It is called at most 4 times during the execution and only if the K-group variables have value greater than '0'.

4. Subroutine AUTCR

'AUTCR' accepts in each call an exponential sequence (generated previously) of variates and transforms it into an autocorrelated sequence. It is called only if any of the Q-group variables equals 1. For each Q-group variable having value 1 it is called M times.

5. Subroutine EARMA

This is the main Subroutine of the program and it has been created so that it can be used for all EARMA(p,q) cases. Because of the generality quite a few parameters are transferred and in each call just one value is returned. This value is the I^{th} arrival or the I^{th} service time as it has been modified because of the moving average. It is called only if the particular EARMA(p,q) case requires moving average; that is only if any of the K-group variables is greater than 0.

6. Time requirements

Some CPU-times have been gathered in the course of the simulation of some EARMA(p,q) cases, using the IBM 360/67, FORTRAN H compiler as follows:

N	M	K-GROUP	QS1	C-GROUP	TIME
2000	500	0	1	0	9 Min. 51 Sec.
5000	500	0	1	0	22 Min. 25 Sec.
10000	500	0	1	0	43 Min. 30 Sec.
10000	500	0	1	CPS=1	55 Min. 20 Sec.

From these results we see that the value of N is the main factor that affects the CPU-time (linearly). We see also that one of the C-group variables may increase the CPU-time by 25%. We ignore the factor of M (it is another main factor as N is) because we may stop the program at any replication, getting the last random number generator seeds and then continuing another time using these seeds.

D. ANALYSIS OF SIMULATION RESULTS

Since no analytical properties of the EARMA(p,q) models can be derived, a simulation has been done to study their properties. But as we have noticed, this program is a general program accomodating all EARMA(p,q) cases. Since our study is limited to only the two cases below, in order to save CPU-time a modified program has been used that simulates these particular two cases.

1. Queue with Dependent Service

The model:

$$\{X_i\} = \mathcal{E}_i$$

$\{S_i\}$ is EARMA(1,0) over $\{E_i, E_{i-1}, \ldots\}$

That is:

$$S_i = \begin{cases} rA_{i-1} & \text{w.p. } r \\ rA_{i-1} + E_i & \text{w.p. } (1-r) \end{cases} \qquad \text{where } A_0 = E_0$$

To simulate this model we chose the values .25, .50, .95, .99 for traffic intensity t and the values .25, .50,

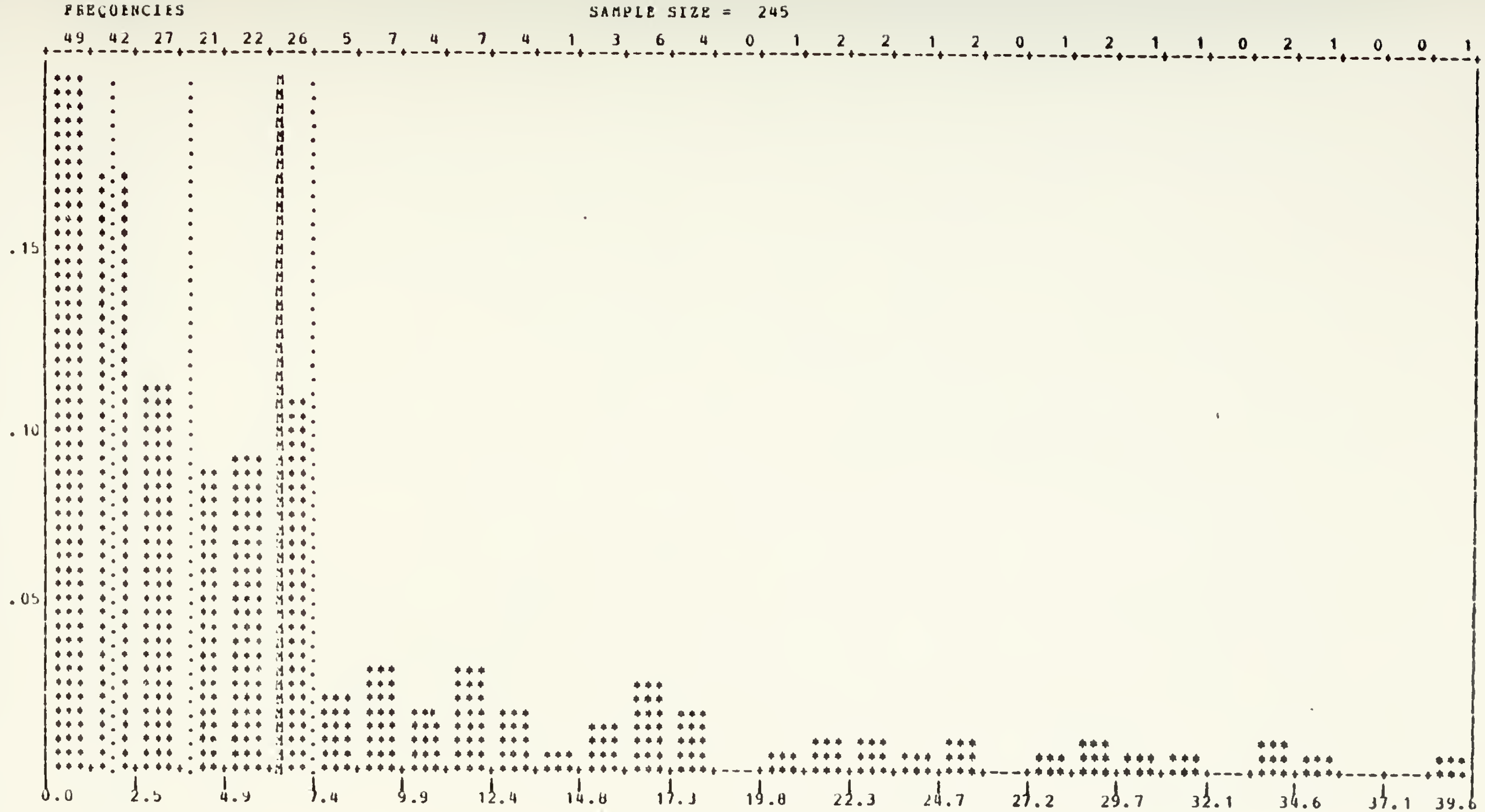
.90, .95, .98 for the correlation r, in order to cover a representative range. In what follows the string S#tt#rr stands for the run with t=tt and r=rr. That is the run S#25#50 stands for the run with t=.25 and r=.50 For each t and r we ran the program and the required statistics were gathered, analyzed and plotted using the Subroutines HISTGS/FS, NCRMPL and EXPLT. The sample statistics from HISTGS/FS were also tabulated separately from the figures so that one could obtain an overall picture of whether or not the waiting times had converged in mean and in distribution to the limiting distribution.

From the analysis of all runs and plots we have the following results:

1. Simulating the type S#25#rr and S#50#rr model (with r =.25, .50 and .90) it was possible for the W and $\bar{W}$ to reach the steady state for N=2000. On the other hand for the type S#25#rr and S#50#rr (with rr = .95, .98) it was necessary to go up to N=10000, in order for the W and $\bar{W}$ to reach the steady state. Thus we see that the high correlation affects the choice of the value of N, requiring N to increase as the correlation is increasing. But not only the correlation affects the choice of N. The traffic intensity affects it much more, since for the type S#95#rr and S#99#rr the W and $\bar{W}$ do not reach the steady state for N=10000 even if r is low. Because of the CPU-time requirements we restricted ourselves to a detailed study of the particular case S#99#98 increasing successively the value of N up to 320,000, where the W and $\bar{W}$ appear to be close to the steady state. Thus we may conclude that high traffic intensity and/or high correlation require a large value of N in order to achieve steady state. Figures 23a through 23d, 24a-24c, 25 and 26a-26d justify the above conclusions. In figures 23a-23d, where we are dealing with

the waiting time of the 2000th (W_{2000}) arrival for the case t=0.50, r=0.25, we can see that W_{2000} has converged to a value of about 6.3, and furthermore a straight line appears in the plot under EXPLT. We also see approximate convergence for the correlated queue for the $\bar{W}_{2000}$, which is presented on figures 28a-28c, and its value has converged to a value of about 3.03 (figure 23d). Note that the introduction of correlation into the service time has increased the average waiting time from 2.5 to about 3.03, or by approximately 20%.

We cannot say the same for figures 24a-24c where, because of the high correlation (r=0.98) we do not have convergence of W and $\bar{W}$, although N=10,000, and for figure 25 where, because of the high traffic intensity (t=0.99) we do not have convergence of W and $\bar{W}$. Figures 26a-26d also show us the non-convergence of W, $\bar{W}$, because of the high correlation and the high traffic intensity (t=0.99, r=0.98). This is the case for which we eventually had to carry the simulation out to about N=500,000 to observe a stationary queue.



```
CENTRAL TENDENCY            SPREAD                      HIGHER CENTRAL MOMENTS     DISTRIBUTION

MEAN        6.365520E 00    VARIANCE    5.483366E 01    M3         8.574727E 02    MINIMUM                    2.400000E-03
MEDIAN      3.676499E 00    STD DEV     7.404976E 00    M4         2.266623E 04    .10 QUANTILE               4.852000E-01
TRIMEAN     4.152908E 00    COEF VAR    1.163295E 00    SKEWNESS   2.111784E 00    .25 QUANTILE  (HINGE)      1.586349E 00
MIDMEAN     4.102381E 00    MEAN DEV    4.728097E 00    KURTOSIS   4.536499E 00    .50 QUANTILE  (MEDIAN)     3.676499E 00
MIDRANGE    1.978464E 01    RANGE       3.956449E 01    BETA1      8.476017E 02    .75 QUANTILE  (HINGE)      7.272300E 00
GEOM MEAN   3.123071E 00    MIDSPREAD   5.685950E 00    BETA2      2.236936E 04    .90 QUANTILE               1.655489E 01
HARM MEAN   3.229375E-01                                                           MAXIMUM                    3.956689E 01
```

Figure 23a - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIME SEQUENCE AND POISSON INPUT. HISTOGRAM OF THE WAITING TIMES W_{2000} WITHOUT ZEROS FROM THE RUN S#50#25; m=500 REPLICATIONS, RX=.2, RS=.4. No OF ZEROS=255.

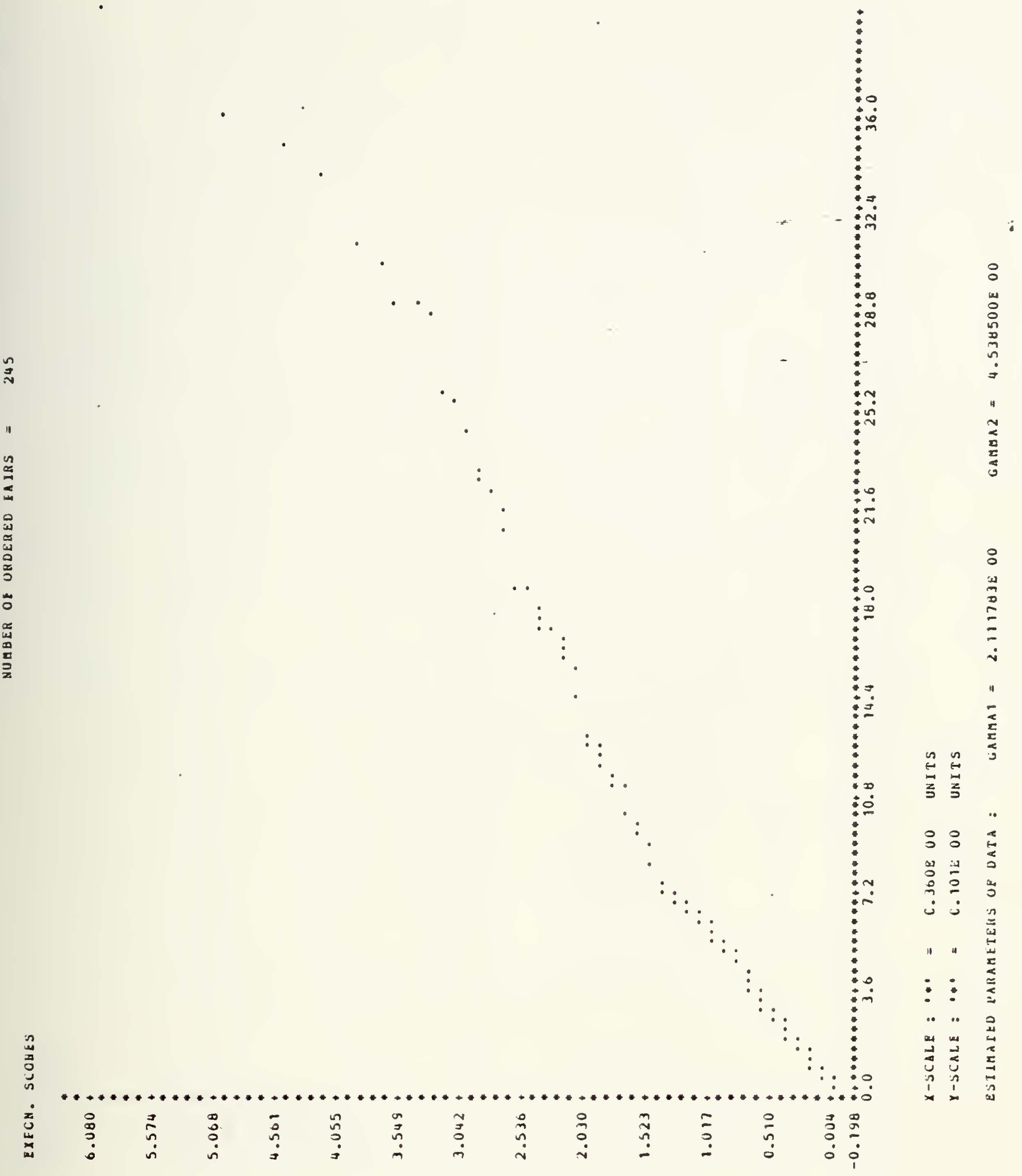


Figure 23b - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIME SEQUENCE AND POISSON INPUT. EXPONENTIAL PLOT (EXPLT) OF THE WAITING TIMES W_{2000} WITHOUT ZEROS FROM THE RUN S#50#25; m=500 REPLICATIONS, RX=.2, RS=.4. No OF ZEROS=255.

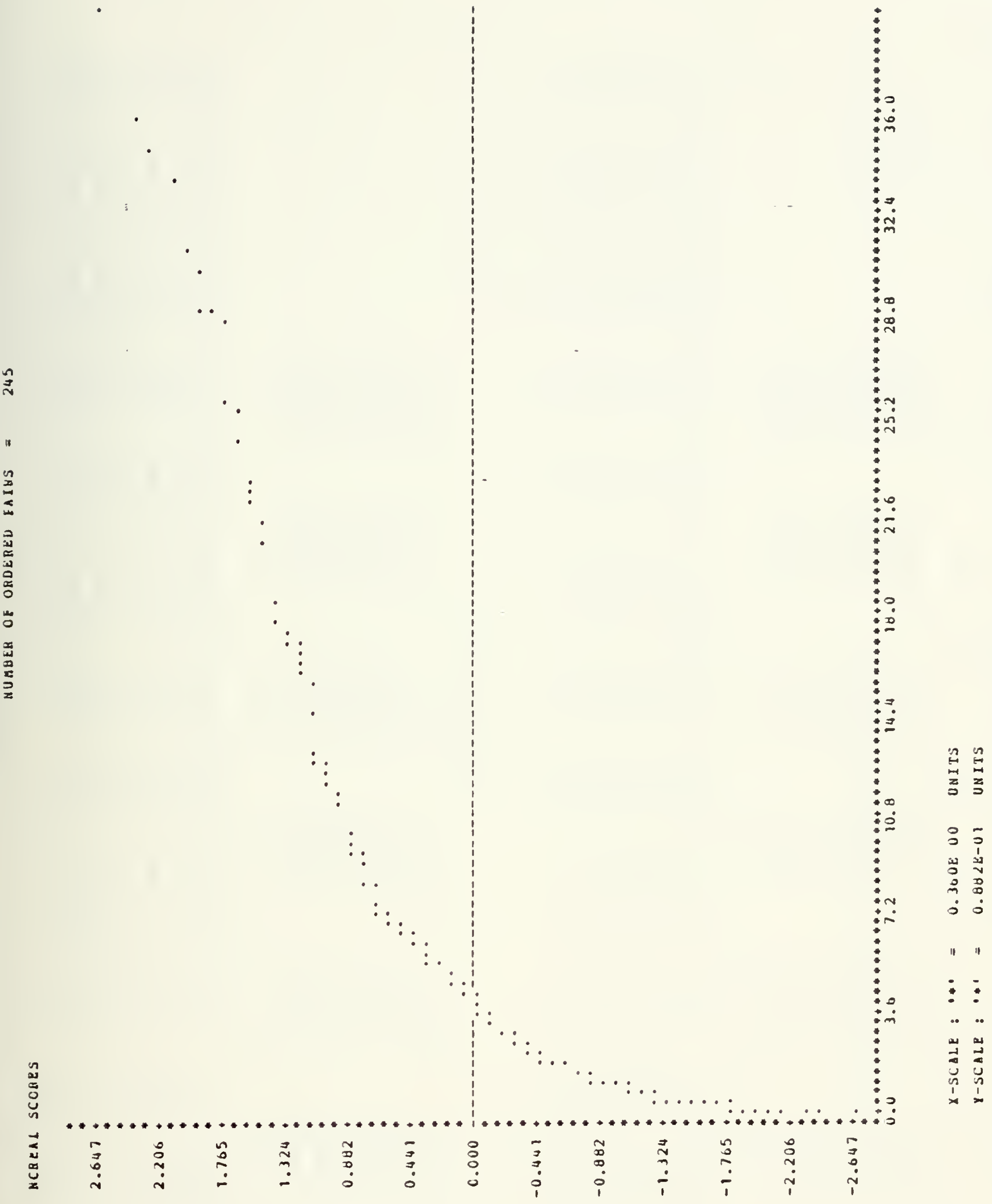
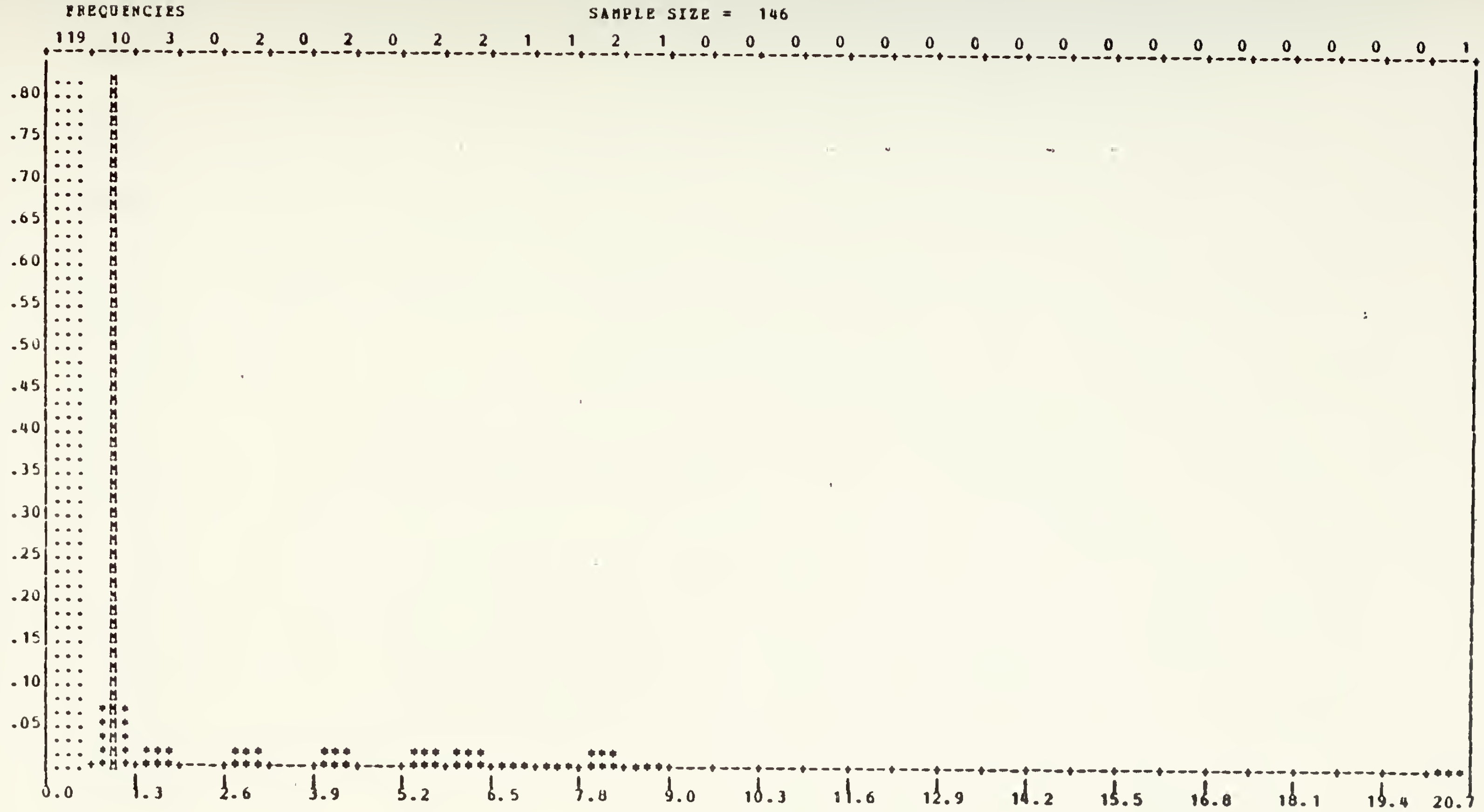


Figure 23c - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIME SEQUENCE AND POISSON INPUT. NORMAL PLOT (NORMPL) OF THE WAITING TIMES W_{2000} WITHOUT ZEROS FROM THE RUN S#50#25; m=500 REPLICATIONS, RX=.2, RS=.4. No OF ZEROS=255.

```
                                                                       CORRELATED QUEUE

 PARMTR       X 2       X 4       S 2       S 4     W 1 + OS     W 1     W 2 + OS     W 2     W 3 + OS     W 3     W 4 + OS     W 4

MEAN       5001.965  9998.750  2498.716  4992.211   3.07758   6.08218   3.06554   6.01085   3.12886   6.28286   3.11910   6.36552
S(MEAN)       7.0361    9.0354    4.5052    6.5871   0.24952   0.41368   0.26895   0.45710   0.24779   0.41009   0.27188   0.47309
ST. DEV.    157.3321  202.0377  100.7388  147.2915   5.57942   6.57991   6.01397   7.29935   5.54076   6.47113   6.07936   7.40498
SKEWNESS     -0.2306   -0.2395    0.0667    0.2586   4.06781   3.55424   3.42195   2.55988   2.43921   1.61747   3.03096   2.11178
KURTOS.       0.1442    0.1382   -0.0233    0.1448  31.44217  24.58147  17.09064  10.08711   6.44455   2.54303  10.37912   4.53850
SMPL SIZE   500.0000     0.0    500.0000  500.0000 500.00000 253.00000 500.00000 255.00000 500.00000 249.00000 500.00000 245.00000

 PARMTR        W 1  BAR        W 2  BAR        W 3  BAR        W 4  BAR          D 2              D 4

MEAN            3.0040035       3.0257206       3.0361691       3.0264702      -0.0032524      -0.0003933
S(MEAN)         0.0384256       0.0288684       0.0229700       0.0199288       0.0082827       0.0060741
ST. DEV.        0.8592227       0.6455169       0.5136244       0.4456213       0.1852060       0.1358213
SKEWNESS        1.2956314       1.0153809       0.8071278       0.7198124       0.1208596       0.0878487
KURTOS.         2.9877872       1.3304682       0.8637352       0.7924871      -0.1567421       0.3080139
SMPL SIZE     500.0000000     500.0000000     500.0000000     500.0000000     500.0000000     500.0000000


                                                                      UNCORRELATED QUEUE

 PARMTR       X 2       X 4       S 2       S 4     W 1 + OS     W 1     W 2 + OS     W 2     W 3 + OS     W 3     W 4 + OS     W 4

MEAN       5001.965  9998.750  2499.844  4999.156   2.50055   5.00109   2.57197   5.14395   2.57820   5.13586   2.72569   5.14280
S(MEAN)       7.0361    9.0354    3.5683    5.1405   0.19253   0.31361   0.21395   0.36101   0.19706   0.31915   0.20248   0.31484
ST. DEV.    157.3321  202.0377   79.7889  114.9458   4.30517   4.95860   4.78403   5.70811   4.40636   5.05628   4.52758   5.12527
SKEWNESS     -0.2306   -0.2395    0.0893    0.1489   2.72664   2.06557   3.80018   3.13885   2.40065   1.65638   2.46332   1.82443
KURTOS.       0.1442    0.1382   -0.0001   -0.0751  10.26487   6.32769  24.89131  17.73781   6.42046   2.84681   6.98296   3.61722
SMPL SIZE   500.0000     0.0    500.0000  500.0000 500.00000 250.00000 500.00000 250.00000 500.00000 251.00000 500.00000 265.00000

 PARMTR        W 1  BAR        W 2  BAR        W 3  BAR        W 4  BAR          D 2              D 4

MEAN            2.4863586       2.4965744       2.5005102       2.4961290      -0.0021236       0.0030810
S(MEAN)         0.0275175       0.0194881       0.0155533       0.0134737       0.0078315       0.0056511
ST. DEV.        0.6153103       0.4357680       0.3477832       0.3012817       0.1751172       0.1263634
SKEWNESS        1.1364145       0.8768911       0.6178530       0.4541365       0.2281407       0.1533930
KURTOS.         1.8650570       0.9444056       0.3653927       0.1611729      -0.0018263       0.5103149
SMPL SIZE     500.0000000     500.0000000     500.0000000     500.0000000     500.0000000     500.0000000
```

Figure 23d - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. TABULATION OF SAMPLE STATISTICS FOR THE DISTRIBUTIONS OF CUMULATED INTERARRIVAL TIMES AT N=1000 AND N=2000 (X 2 AND X 4), CUMULATED SERVICE TIMES (S 2 AND S 4), WAITING TIMES, WITH OR WITHOUT ZEROS (WI + 0, WI, I FOR CASE N=500, 1000, 1500, 2000), CUMULATED WAITING TIMES (W1 BAR etc.) AND THE AVERAGED DIFFERENCES BETWEEN S2 AND X2 (D2); OBTAINED FROM m=500 REPLICATIONS OF THE RUN S#50#25; EX=.2; RS=.4 .



```
CENTRAL TENDENCY            SPREAD                       HIGHER CENTRAL MOMENTS     DISTRIBUTION

MEAN        8.810984E-01    VARIANCE    5.797456E 00     M3          7.037097E 01   MINIMUM                        3.600000E-03
MEDIAN      1.373500E-01    STD DEV     2.407790E 00     M4          1.188605E 03   .10 QUANTILE                   1.040000E-02
TRIMEAN     1.668000E-01    COEF VAR    2.732714E 00     SKEWNESS    5.041243E 00   .25 QUANTILE  (HINGE)          3.660000E-02
MIDMEAN     1.499310E-01    MEAN DEV    8.336134E-01     KURTOSIS    3.236411E 01   .50 QUANTILE  (MEDIAN)         1.373500E-01
MIDRANGE    1.033659E 01    RANGE       2.066599E 01     BETA1       6.893161E 01   .75 QUANTILE  (HINGE)          3.559000E-01
GEOM MEAN   1.371452E-01    MIDSPREAD   3.192999E-01     BETA2       1.157410E 03   .90 QUANTILE                   1.538699E 00
HARM MEAN   3.280574E-02                                                            MAXIMUM                        2.066959E 01
```

Figure 24a - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIME SEQUENCE AND POISSON INPUT. HISTOGRAM OF THE WAITING TIMES W_{10000} WITHOUT ZEROS FROM THE RUN S#25#98; m=500 REPLICATIONS, RX=2.5, RS=10; No OF ZEROS=354.

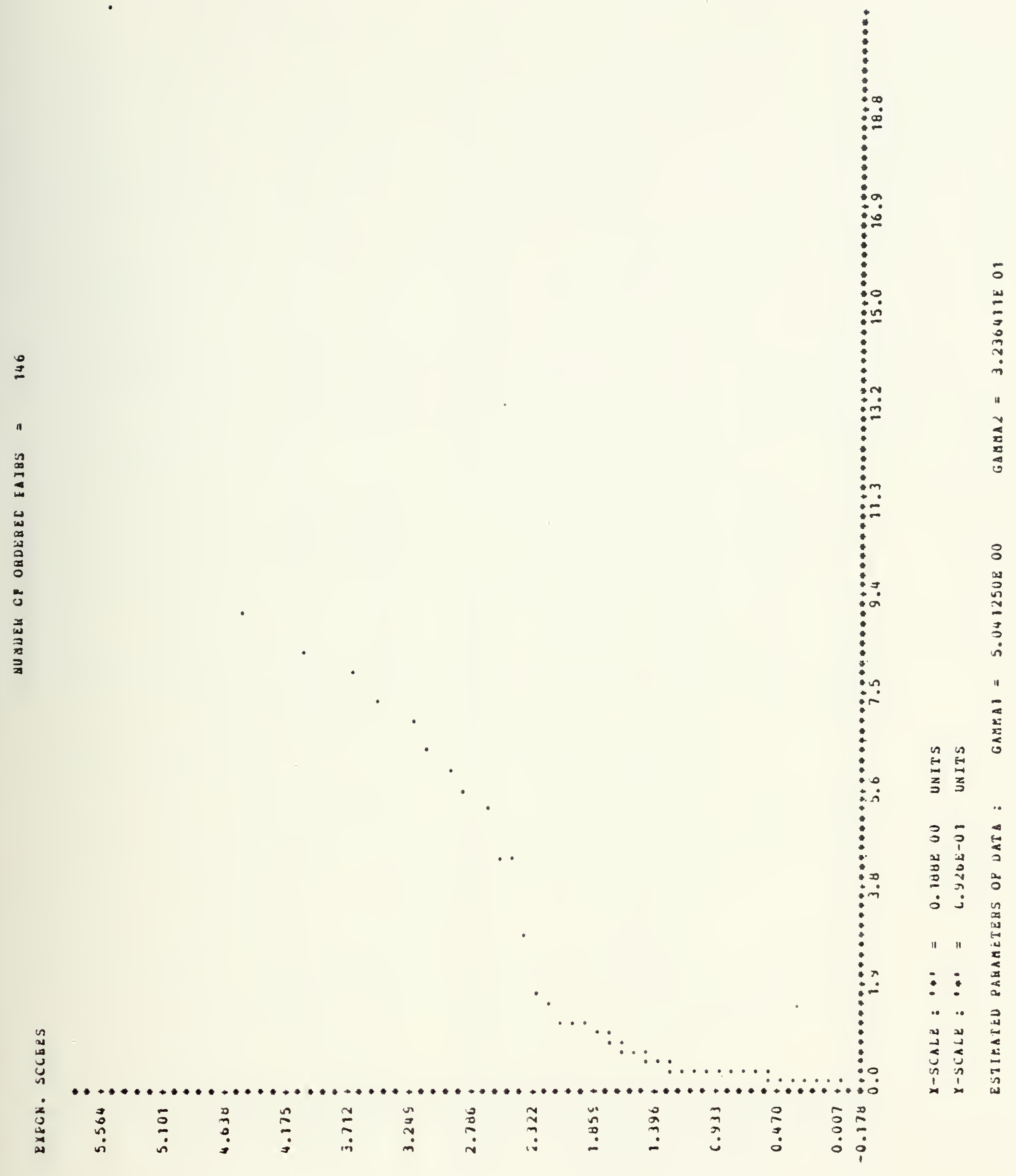


Figure 24b - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIME SEQUENCE AND POISSON INPUT. EXPONENTIAL PLOT (EXPLT) OF THE WAITING TIMES W_{10000} WITHOUT ZEROS FROM THE RUN S#25#98; m=500 REPLICATIONS, RX=2.5, RS=10; No OF ZEROS=354.

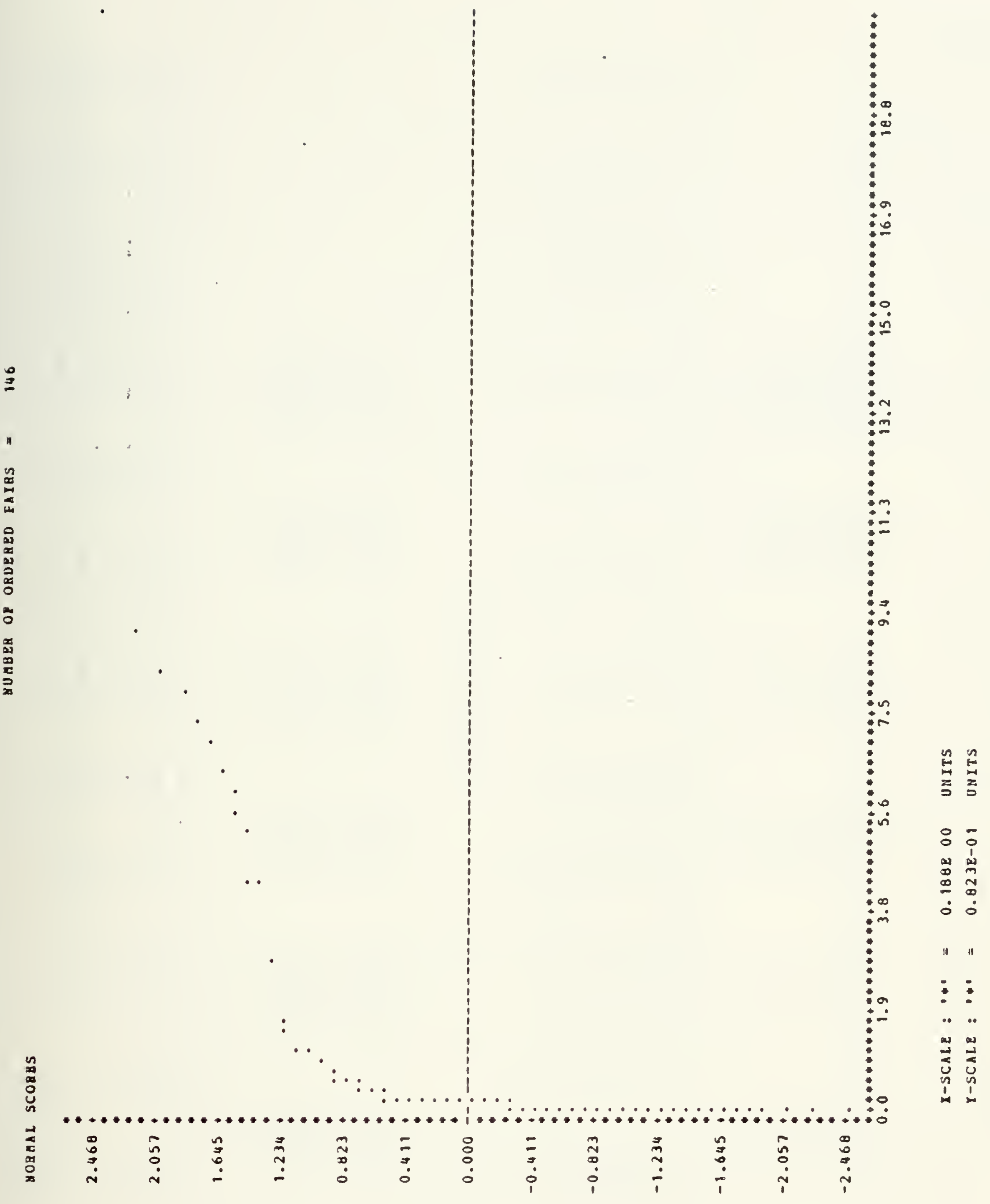
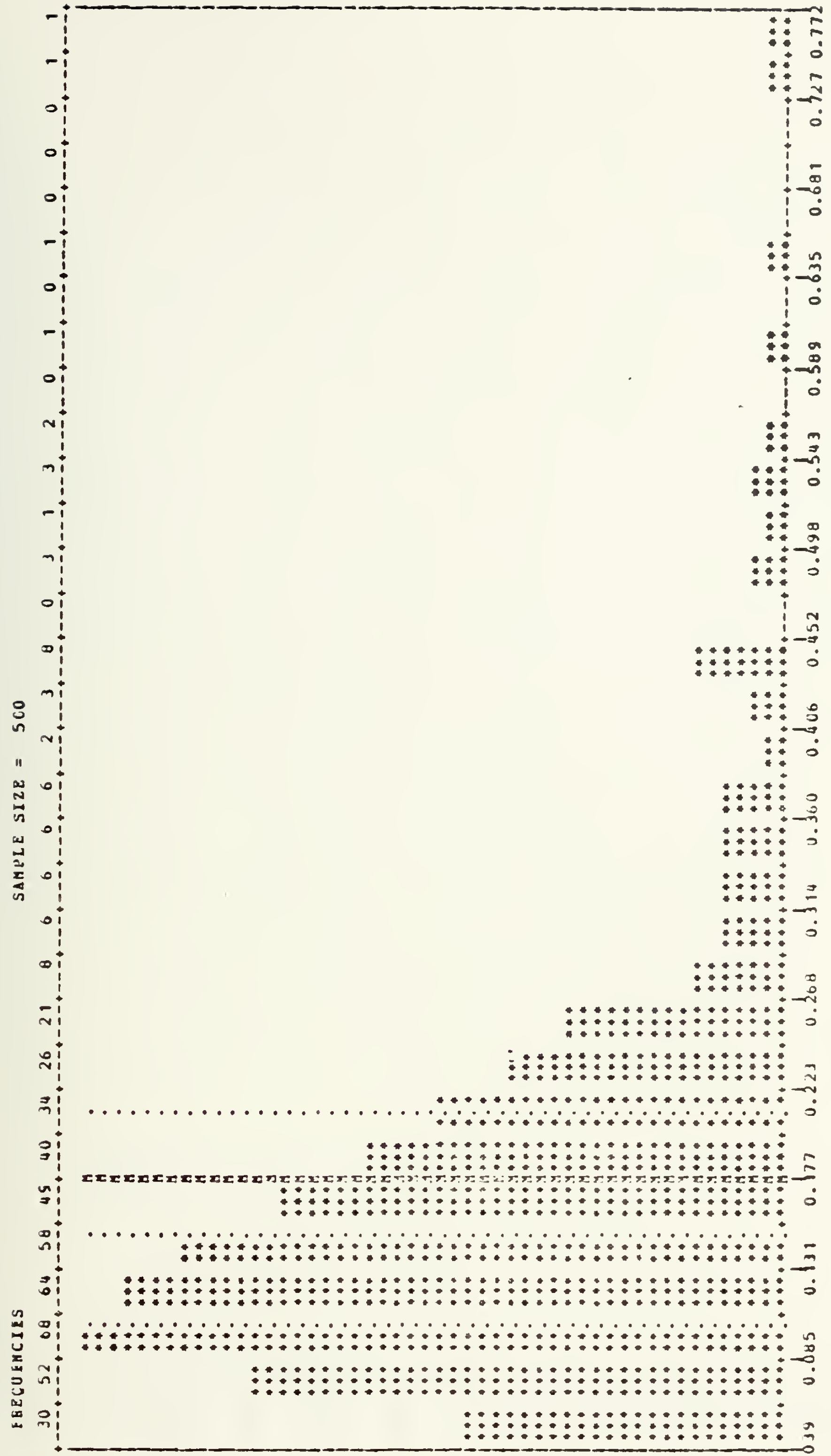


Figure 24c - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIME SEQUENCE AND POISSON INPUT. NORMAL PLOT (NORMPL) OF THE WAITING TIMES W_{10000} WITHOUT ZEROS FROM THE RUN S#25#98; m=500 REPLICATIONS, RX=2.5, RS=10; No OF ZEROS=354.

```
                                                     CORRELATED QUEUE

 PARMTS       X 2        X 4        S 2        S 4     W 1 + OS     W 1      W 2 + OS     W 2      W 3 + OS     W 3      W 4 + OS     W 4

MEAN       1999.496   3999.234   495.382    993.617    0.18153    0.78246    0.10872    0.42469    0.11847    0.59834    0.25728    0.88110
S(MEAN)       1.2384     1.7382     2.9624     4.3755    0.05074    0.20991    0.02380    0.08739    0.03447    0.16622    0.06075    0.19927
ST. DEV.     27.6907    38.8671    66.2412    97.8387    1.13456    2.26080    0.53219    0.96873    0.77081    1.65365    1.35847    2.40779
SKEWNESS     -0.0261    -0.2654     0.2060     0.2402   11.70774    5.67824    9.35595    4.79691   11.72616    5.23263    9.36337    5.04124
KURTOS.       0.1781     0.1013    -0.0917     0.1235  159.47882   36.05627  101.97438   25.38043  155.19870   29.24605  113.06558   32.36411
SMPL SIZE   500.0000   500.0000   500.0000   500.0000  500.00000  116.00000  500.00000  128.00000  500.00000   99.00000  500.00000  146.00000

 PARMTS        W 1  BAR       W 2  BAR       W 3  BAR       W 4  BAR         D 2           D 4

MEAN          0.1663668      0.1690013      0.1701906      0.1707705     -0.0008230    -0.0005616
S(MEAN)       0.0083418      0.0068714      0.0056008      0.0047813      0.0006433     0.0004592
ST. DEV.      0.1865285      0.1536493      0.1252386      0.1069135      0.0143855     0.0102669
SKEWNESS      3.7813730      3.3361349      2.6342467      2.0241365      0.1306823     0.2371868
KURTOS.      23.4013367     17.2875519     10.3973818      5.6284195     -0.1873236     0.1636181
SMPL SIZE   500.0000000    500.0000000    500.0000000    500.0000000    500.0000000   500.0000000


                                                     UNCORRELATED QUEUE

 PARMTS       X 2        X 4        S 2        S 4     W 1 + OS     W 1      W 2 + OS     W 2      W 3 + OS     W 3      W 4 + OS     W 4

MEAN       1999.496   3999.234   499.456    998.720    0.02767    0.11249    0.02637    0.11986    0.04066    0.16530    0.04013    0.14645
S(MEAN)       1.2384     1.7382     0.3155     0.4621    0.00305    0.00873    0.00330    0.01112    0.00533    0.01742    0.00432    0.01165
ST. DEV.     27.6907    38.8671     7.0557    10.3337    0.06815    0.09685    0.07377    0.11664    0.11920    0.19324    0.09664    0.13631
SKEWNESS     -0.0261    -0.2654     0.2372     0.0944    3.11780    1.21369    4.00196    1.76561    4.78121    2.43433    3.59277    1.95713
KURTOS.       0.1781     0.1013     0.0277     0.0293   10.78555    1.41914   19.01723    3.32073   29.88892    7.72525   17.81947    6.29365
SMPL SIZE   500.0000   500.0000   500.0000   500.0000  500.00000  123.00000  500.00000  110.00000  500.00000  123.00000  500.00000  137.00000

 PARMTS        W 1  BAR       W 2  BAR       W 3  BAR       W 4  BAR         D 2           D 4

MEAN          0.0332546      0.0332368      0.0332776      0.0332694     -0.0000081    -0.0000513
S(MEAN)       0.0001379      0.0000957      0.0000833      0.0000733      0.0002550     0.0001804
ST. DEV.      0.0030838      0.0021391      0.0018631      0.0016391      0.0057017     0.0040337
SKEWNESS      0.4139023      0.3522996      0.2679055      0.3216345      0.1141537     0.2426357
KURTOS.       0.4359894      0.0770941     -0.0173473     -0.0419693      0.0508137    -0.0223646
SMPL SIZE   500.0000000    500.0000000    500.0000000    500.0000000    500.0000000   500.0000000
```

Figure 24d - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. TABULATION OF SAMPLE STATISTICS FOR THE DISTRIBUTIONS OF CUMULATED INTERARRIVAL TIMES AT N=5000 AND N=10000 (X 2 AND X 4), CUMULATED SERVICE TIMES (S 2 AND S 4), WAITING TIMES, WITH OR WITHOUT ZEROS (WI + 0, WI, I FOR CASE N=2500, 5000, 7500, 10000), CUMULATED WAITING TIMES (W1 BAR etc.) AND THE AVERAGED DIFFERENCES BETWEEN S2 AND X2 (D2); OBTAINED FROM m=500 REPLICATIONS OF THE RUN S#25#9E; RX=2.5; RS=10 .



```
CENTRAL TENDENCY                SPREAD                           HIGHER CENTRAL MOMENTS        DISTRIBUTION

MEAN       1.707705E-01         VARIANCE   1.143050E-02          M3         2.473645E-03       MINIMUM                       3.940000E-02
MEDIAN     1.420500E-01         STD DEV    1.069135E-01          M4         1.127358E-03       .10 QUANTILE                  7.165998E-02
TRIMEAN    1.474999E-01         COEF VAR   6.260653E-01          SKEWNESS   2.024137E 00       .25 QUANTILE  (HINGE)         9.859997E-02
MIDMEAN    1.471515E-01         MEAN DEV   7.258219E-02          KURTOSIS   5.628420E 00       .50 QUANTILE  (MEDIAN)        1.420500E-01
MIDRANGE   4.059500E-01         RANGE      7.330999E-01          BETA1      2.456825E-03       .75 QUANTILE  (HINGE)         2.072999E-01
GEOM MEAN  1.459890E-01         MIDSPREAD  1.087000E-01          BETA2      1.119903E-03       .90 QUANTILE                  2.913000E-01
HARM MEAN  1.265051E-01                                                                        MAXIMUM                       7.725000E-01
```

Figure 24e - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. HISTOGRAM OF THE CUMULATED AND AVERAGED WAITING TIMES $\bar{W}_{10000}$ FROM THE RUN S#25#98; m=500 REPLICATIONS, RX=2.5, RS=10 .

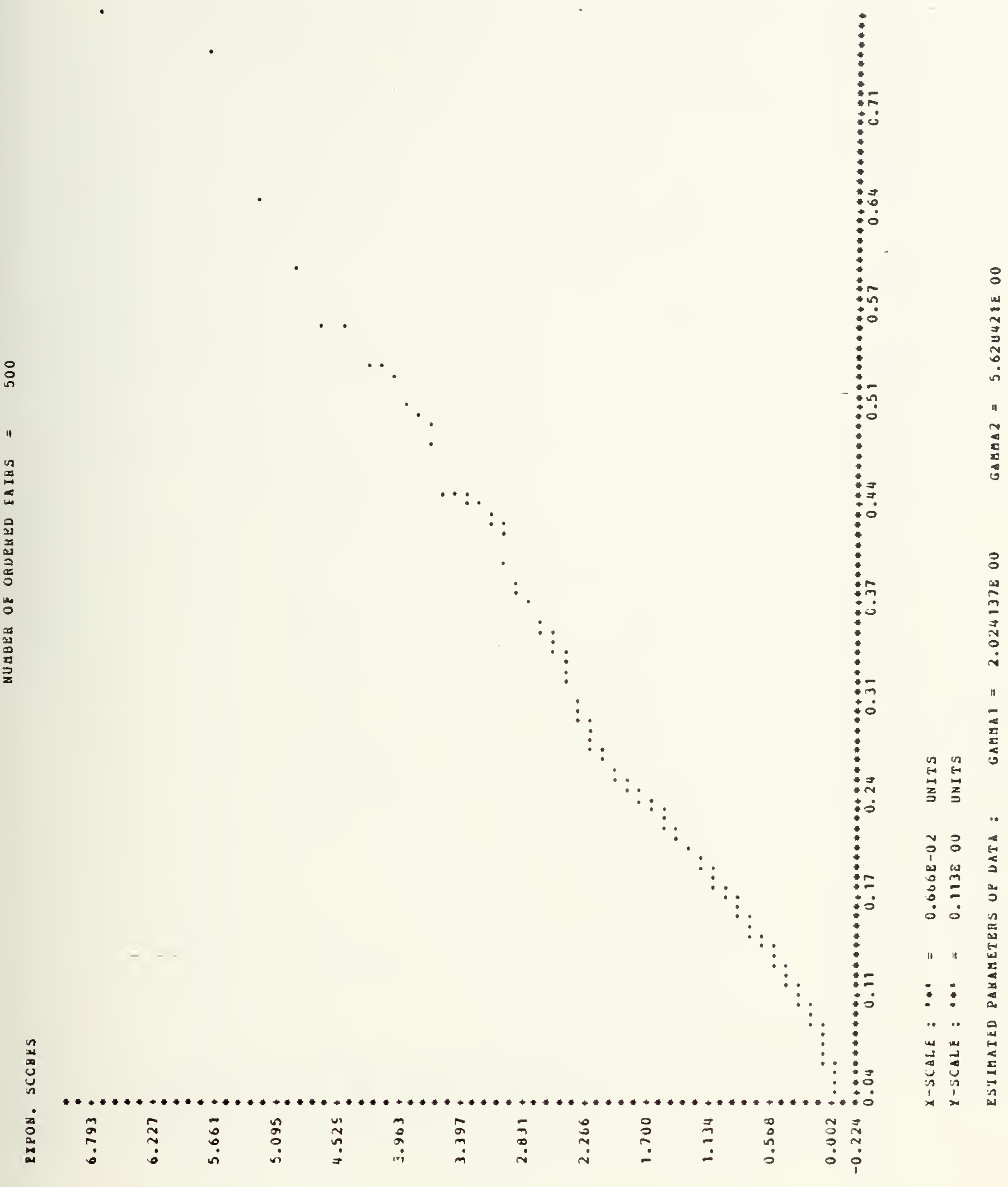


Figure 24f - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. EXPONENTIAL PLOT (EXPLT) OF THE CUMULATED AND AVERAGED WAITING TIMES $\overline{W}_{10000}$ FROM THE RUN S#25#98; m=500 REPLICATIONS, RX=2.5, RS=10 .

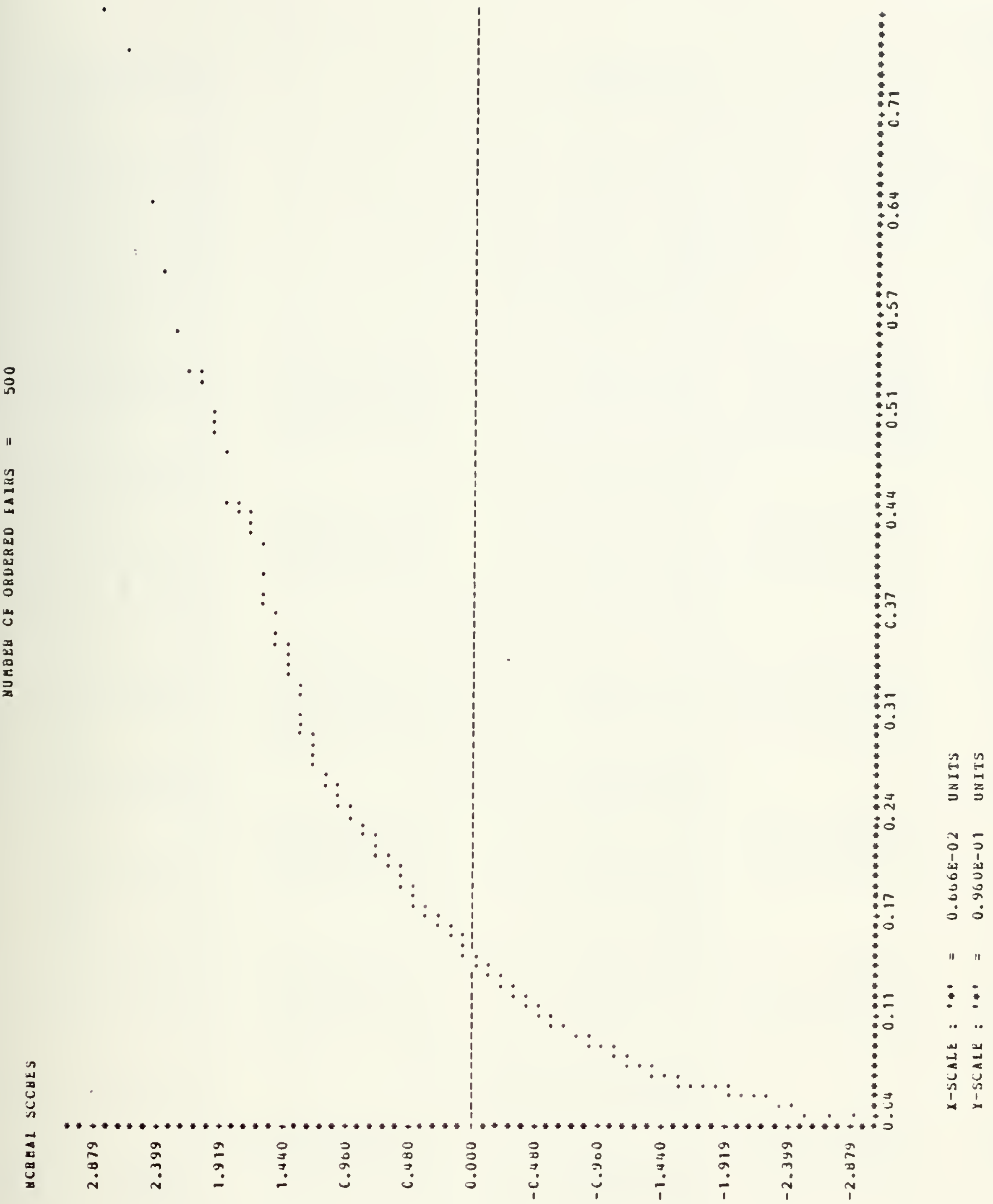


Figure 24g - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. NORMAL PLOT (NORMPL) OF THE CUMULATED AND AVERAGED WAITING TIMES $\bar{W}_{10000}$ FROM THE RUN S#25#98; m=500 REPLICATIONS, RX=2.5, RS=10 .

CORRELATED QUEUE

PARMTR	X 2	X 4	S 2	S 4	W 1 + OS	W 1	W 2 + OS	W 2	W 3 + OS	W 3	W 4 + OS	W 4
MEAN	1683.007	3366.192	1664.411	3330.068	16.96959	17.24553	22.27824	22.59456	25.56830	25.96404	28.15018	28.78342
S(MEAN)	1.0424	1.4631	1.2589	1.8697	0.63766	0.64054	0.82674	0.82978	1.02681	1.03253	1.14832	1.15816
ST. DEV.	23.3062	32.7162	29.0446	41.8077	14.25894	14.20781	18.48639	18.42409	22.96030	22.91158	25.67712	25.61078
SKEWNESS	-0.0261	-0.2053	-0.0354	-0.0217	1.09583	1.09519	1.18767	1.19065	1.42630	1.42657	1.75003	1.75914
KURTOS.	0.1783	0.1013	-0.1659	-0.1877	0.82240	0.82057	1.30003	1.30672	2.64276	2.65312	4.60946	4.65198
SMPL SIZE	500.0000	500.0000	500.0000	500.0000	500.00000	492.00000	500.00000	493.00000	500.00000	492.00000	500.00000	489.00000

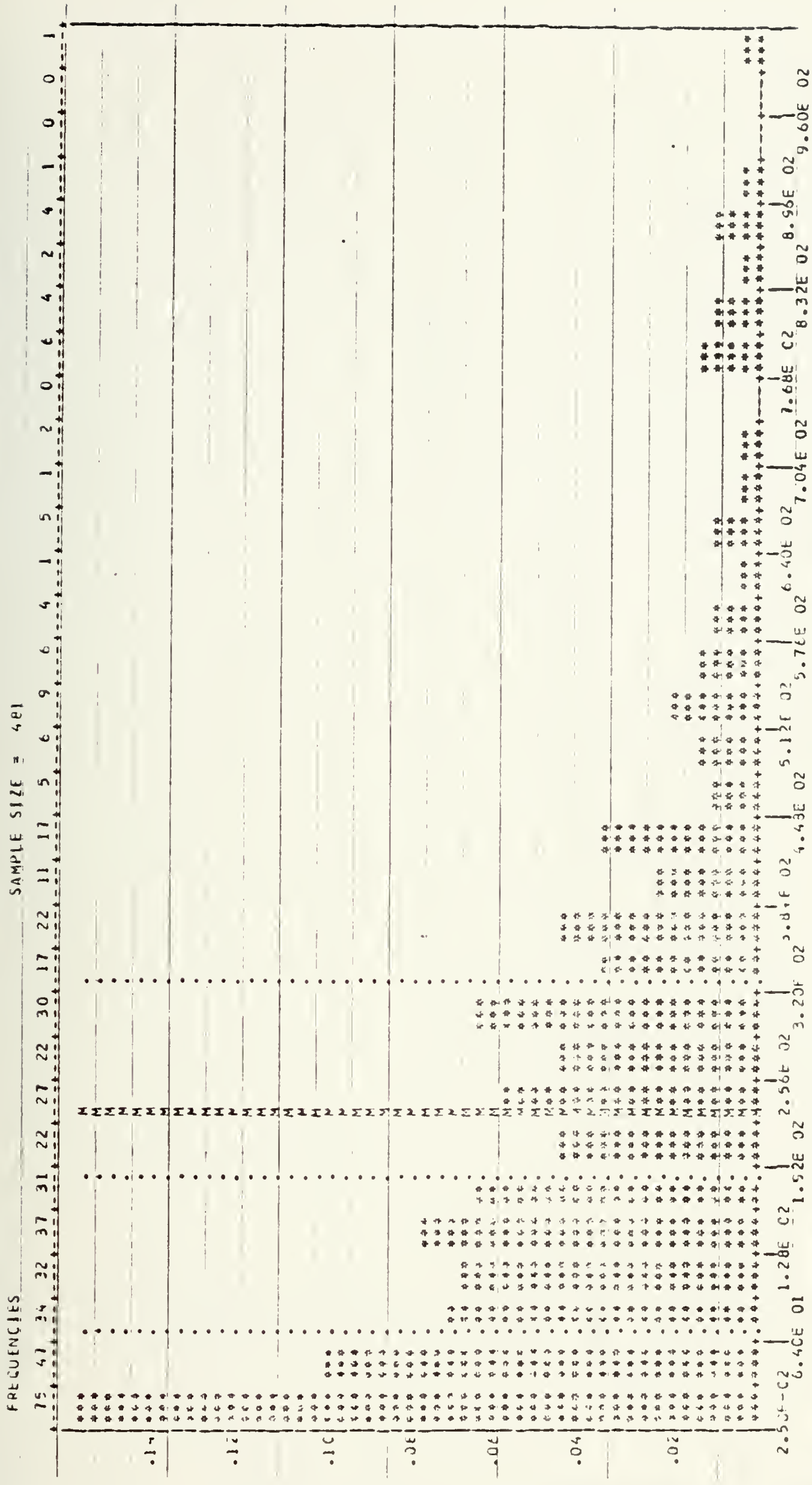
PARMTR	W 1 BAR	W 2 BAR	W 3 BAR	W 4 BAR	D 2	D 4
MEAN	11.8439808	15.9920397	18.7871857	20.7761688	-0.0003522	-0.0002452
S(MEAN)	0.3093749	0.4183092	0.4840423	0.5419208	0.0003232	0.0002345
ST. DEV.	6.9178371	9.3536816	10.8235188	12.1177225	0.0072230	0.0052435
SKEWNESS	1.4603024	1.3524857	1.4713373	1.7270746	0.0722044	0.0792252
KURTOS.	2.2101507	1.5983324	2.2263470	3.9483662	-0.0658259	0.0903988
SMPL SIZE	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000

UNCORRELATED QUEUE

PARMTR	X 2	X 4	S 2	S 4	W 1 + OS	W 1	W 2 + OS	W 2	W 3 + OS	W 3	W 4 + OS	W 4
MEAN	1683.007	3366.192	1665.384	3331.023	14.82400	15.03446	18.82251	19.12897	21.96606	22.27795	23.66727	23.90634
S(MEAN)	1.0424	1.4631	1.0524	1.5408	0.56661	0.56925	0.75766	0.76221	0.91403	0.91937	1.02615	1.03093
ST. DEV.	23.3062	32.7162	23.5316	34.4538	12.67428	12.63930	16.94191	16.90674	20.43629	20.41335	22.94553	22.93684
SKEWNESS	-0.0261	-0.2053	0.2369	0.0945	1.14059	1.13940	1.44110	1.44187	1.34763	1.34367	1.90339	1.90468
KURTOS.	0.1763	0.1018	0.0275	0.0294	0.79820	0.79072	2.37382	2.37581	1.81030	1.80023	5.34098	5.34651
SMPL SIZE	500.0000	500.0000	500.0000	500.0000	500.00000	493.00000	500.00000	492.00000	500.00000	493.00000	500.00000	495.00000

PARMTR	W 1 BAR	W 2 BAR	W 3 BAR	W 4 BAR	D 2	D 4
MEAN	10.3916206	13.8082275	16.1548462	17.7791595	-0.0001574	-0.0001499
S(MEAN)	0.2797736	0.3830756	0.4490776	0.5010985	0.0002948	0.0002138
ST. DEV.	6.2559299	8.5658340	10.0416851	11.2049074	0.0065921	0.0047798
SKEWNESS	1.4325485	1.4735689	1.6260328	1.7394295	0.2896590	0.2765061
KURTOS.	1.9141417	2.0772343	2.9112806	3.5985595	-0.1195488	0.1787348
SMPL SIZE	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000

Figure 25 - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. TABULATION OF SAMPLE STATISTICS FOR THE DISTRIBUTIONS OF CUMULATED INTERARRIVAL TIMES AT N=5000 AND N=10000 (X 2 AND X 4), CUMULATED SERVICE TIMES (S 2 AND S 4), WAITING TIMES, WITH OR WITHOUT ZEROS (WI + 0, WI, I FOR CASE N=2500, 5000, 7500, 10000), CUMULATED WAITING TIMES (W1 BAR etc.) AND THE AVERAGED DIFFERENCES BETWEEN S2 AND X2 (D2); OBTAINED FROM m=500 REPLICATIONS OF THE RUN S#99#25; RX=2.97; RS=3 .



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CENTRAL TENDENCY              SPREAD                        HIGHER CENTRAL MOMENTS        DISTRIBUTION

MEAN        2.258247E 02      VARIANCE    4.109648E 04      M3          1.046672E 07      MINIMUM                     2.500000E-02
MEDIAN      1.763659E 02      STD DEV     2.027227E 02      M4          7.495721E 09      .10 QUANTILE                1.522320E 01
TRIMEAN     1.852640E 02      COEFF VAR   8.976992E-01      SKEWNESS    1.256320E 00      .25 QUANTILE  (HINGE)       6.274524E 01
MIDMEAN     1.647635E 02      MEAN DEV    1.540134E 02      KURTOSIS    1.438172E 00      .50 QUANTILE  (MEDIAN)      1.763659E 02
MIDRANGE    5.120264E 02      RANGE       1.024003E 03      BETA1       1.040154E 07      .75 QUANTILE  (HINGE)       3.256550E 02
GEOM MEAN   1.115239E 02      MIDSPREAD   2.629098E 02      BETA2       7.454352E 09      .90 QUANTILE                4.960342E 02
HARM MEAN   3.725702E 00                                                                  MAXIMUM                     1.024028E 03
```

Figure 26a - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. HISTOGRAM OF THE WAITING TIMES W_{10000} WITHOUT ZEROS FROM THE S#99#98; m=500 REPLICATIONS, RX=2.97; RS=3 ; No OF ZEROS=19.

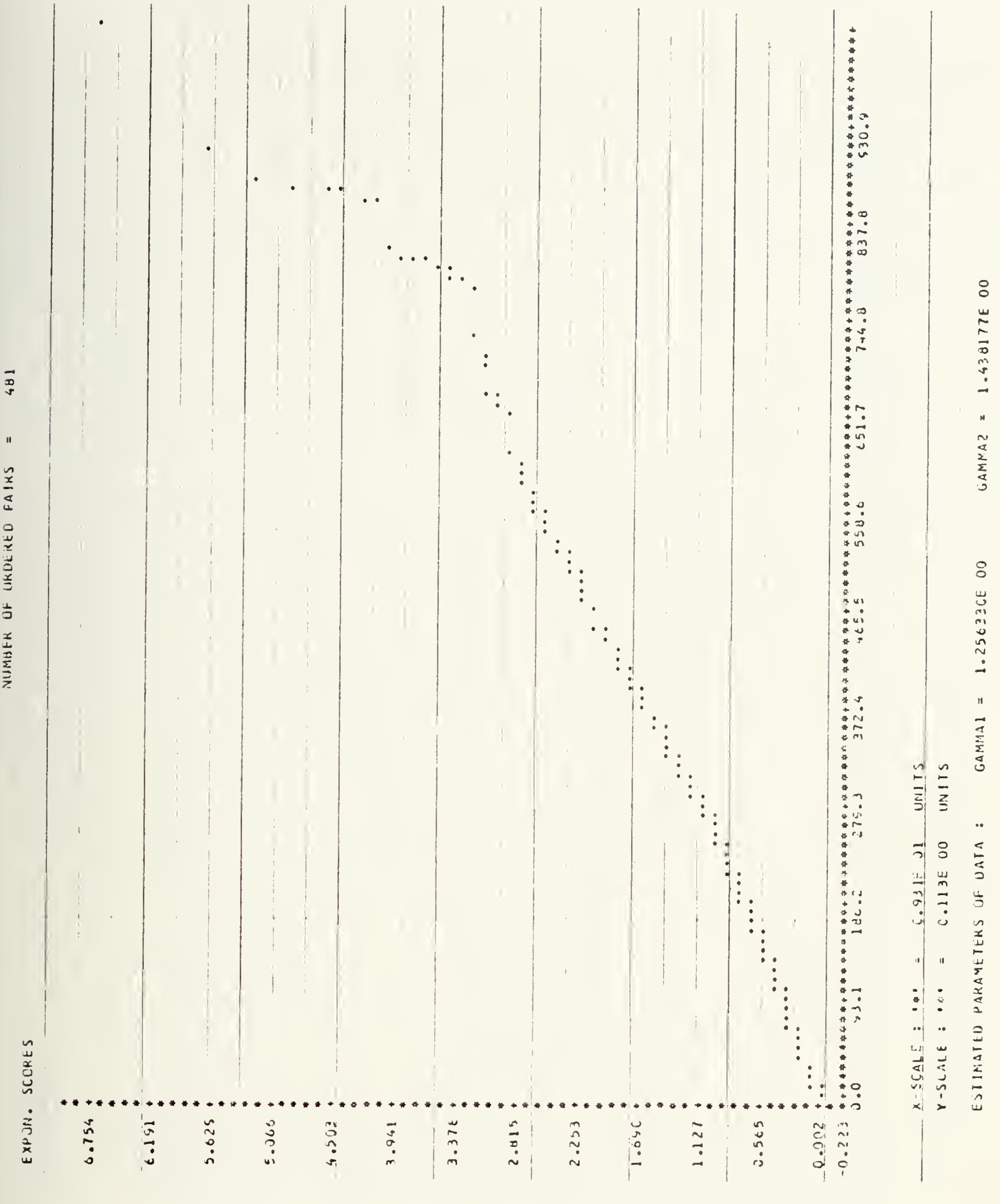


Figure 26b - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. EXPONENTIAL PLOT (EXPLT) OF THE WAITING TIMES W_{10000} WITHOUT ZEROS FROM THE RUN S#99#98; m=500 REPLICATIONS, RX=2.97; RS=3 ; No OF ZEROS=19.

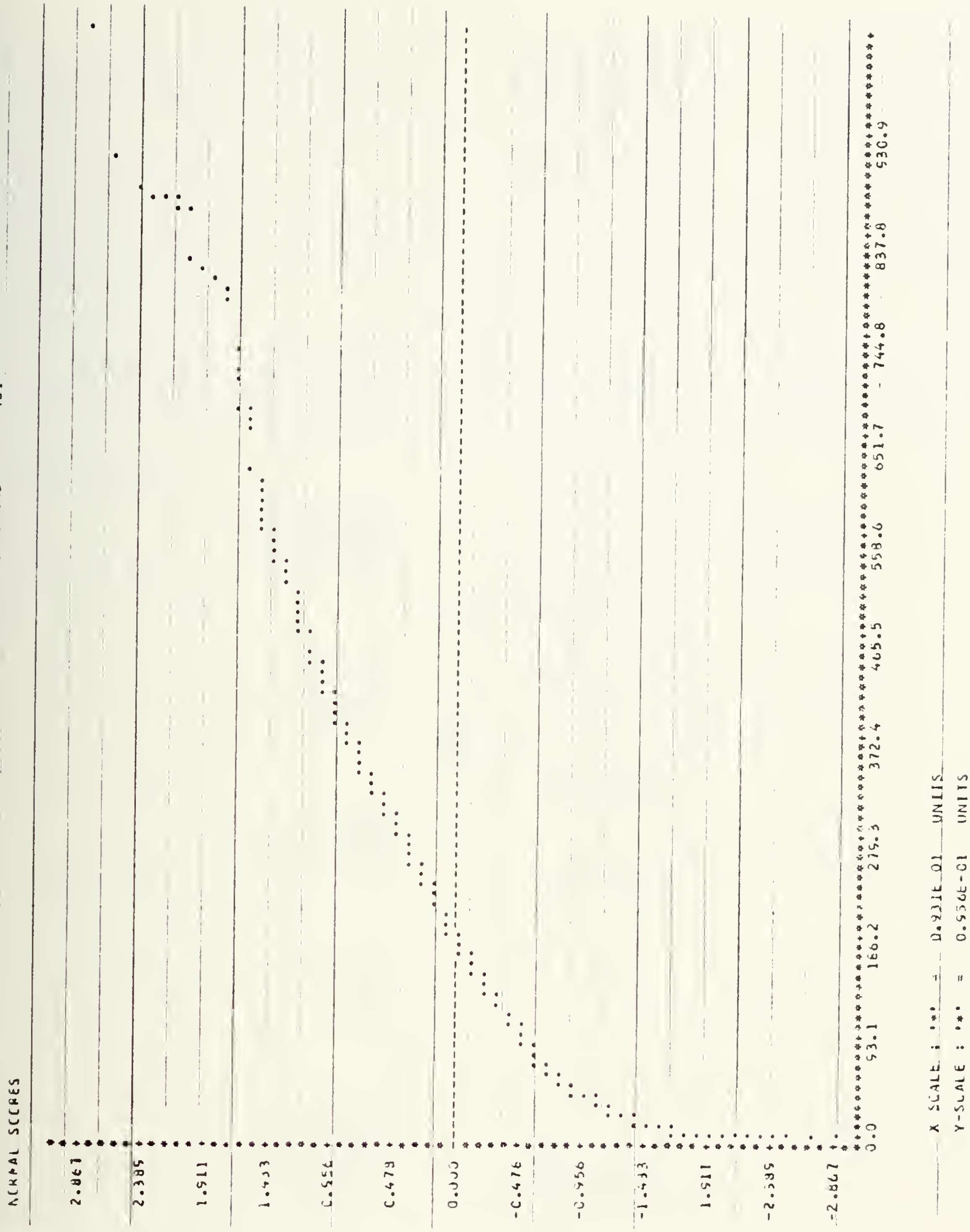


Figure 26c - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. NORMAL PLOT (NORMPL) OF THE WAITING TIMES W_{10000} WITHOUT ZEROS FROM THE RUN S#99#98; m=500 REPLICATIONS, RX=2.97; RS=3 ; No OF ZEROS=19.

```
                                              CORRELATED QUEUE
-------------------------------------------------------------------------------------------------------------------

 PARMTR        X 2       X 4       S 2       S 4    W 1 + OS      W 1    W 2 + OS      W 2    W 3 + OS      W 3    W 4 + OS      W 4

MEAN       1683.007  3365.192  1651.803  3314.000 111.15344 120.55688 144.63533 153.54070 188.04311 159.19820 217.24336 225.82469
S(MEAN)      1.0424    1.4631    9.8800   14.5871   4.63631   4.77766   6.03042   6.17073   7.62512   7.76045   9.09541   9.24335
ST. DEV.    23.3082   32.7162  220.9243  326.1775 103.67105 102.58067 134.84439 133.92052 170.50285 169.03450 203.45904 202.72266
SKEWNESS    -0.0261   -0.2053    0.1554    0.2594   1.15324   1.10072   1.26989   1.25410   1.26027   1.25370   1.26990   1.25633
KURTOS.      0.1783    0.1018    0.0921    0.1221   1.00631   0.96592   1.51814   1.48736   1.80714   1.82495   1.46775   1.45817
SMPL SIZE  500.0000  500.0000  500.0000  500.0000 500.00000 461.00000 500.00000 471.00000 500.00000 472.00000 500.00000 461.00000

 PARMTR        W 1  BAR       W 2  BAR       W 3  BAR       W 4  BAR         D 2            D 4

MEAN         65.8666272     97.8470517    121.9977417    142.2246246     -0.0028737     -0.0018523
S(MEAN)       2.5285978      3.1955700      3.7718716      4.4213009      0.0019880      0.0014556
ST. DEV.     55.5411987     71.4551344     84.3416443     98.8633423      0.0444540      0.0325435
SKEWNESS      1.5251767      1.3165245      1.3510013      1.4051094      0.1860167      0.2517110
KURTOS.       2.5743933      1.7875967      1.6947069      2.0297928     -0.1270666      0.1280117
SMPL SIZE   500.0000000    500.0000000    500.0000000    500.0000000    500.0000000    500.0000000
```

```
                                             UNCORRELATED QUEUE
-------------------------------------------------------------------------------------------------------------------
 PARMTR        X 2       X 4       S 2       S 4    W 1 + OS      W 1    W 2 + OS      W 2    W 3 + OS      W 3    W 4 + OS      W 4

MEAN       1683.007  3365.192  1665.384  3331.023  14.82400  15.03446  18.82291  19.12897  21.96606  22.27795  23.66727  23.90634
S(MEAN)      1.0424    1.4631    1.0524    1.5406   0.56681   0.56925   0.75766   0.76221   0.91403   0.91937   1.02615   1.03093
ST. DEV.    23.3082   32.7162   23.5316   34.4538  12.67428  12.63930  16.94151  16.90674  20.43825  20.41335  22.94555  22.93684
SKEWNESS    -0.0261   -0.2053    0.2369    0.0945   1.14059   1.13940   1.44110   1.44187   1.34763   1.34367   1.90335   1.90468
KURTOS.      0.1783    0.1018    0.0275    0.0294   0.79820   0.79072   2.37382   2.37581   1.81030   1.80023   5.34098   5.34691
SMPL SIZE  500.0000  500.0000  500.0000  500.0000 500.00000 493.00000 500.00000 492.00000 500.00000 493.00000 500.00000 495.00000
-------------------------------------------------------------------------------------------------------------------
 PARMTR        W 1  BAR       W 2  BAR       W 3  BAR       W 4  BAR         D 2            D 4

MEAN         10.3916206     13.8082275     16.1548462     17.7791595     -0.0001574     -0.0001499
S(MEAN)       0.2797736      0.3830756      0.4490776      0.5010985      0.0002948      0.0002138
ST. DEV.      6.2559259      8.5658340     10.0416851     11.2049074      0.0065921      0.0047798
SKEWNESS      1.4325485      1.4735689      1.6260328      1.7394295      0.2896590      0.2765061
KURTOS.       1.9141417      2.0772343      2.9112806      3.5989599     -0.1195488      0.1787348
SMPL SIZE   500.0000000    500.0000000    500.0000000    500.0000000    500.0000000    500.0000000
```

Figure 26d - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. TABULATION OF SAMPLE STATISTICS FOR THE DISTRIBUTIONS OF CUMULATED INTERARRIVAL TIMES AT N=5000 AND N=10000 (X 2 AND X 4), CUMULATED SERVICE TIMES (S 2 AND S 4), WAITING TIMES, WITH OR WITHOUT ZEROS (WI + 0, WI, I FOR CASE N=2500, 5000, 7500, 10000), CUMULATED WAITING TIMES (W1 BAR etc.) AND THE AVERAGED DIFFERENCES BETWEEN S2 AND X2 (D2); OBTAINED FROM m=500 REPLICATIONS OF THE RUN S#99#98; RX=2.97; RS=3 .

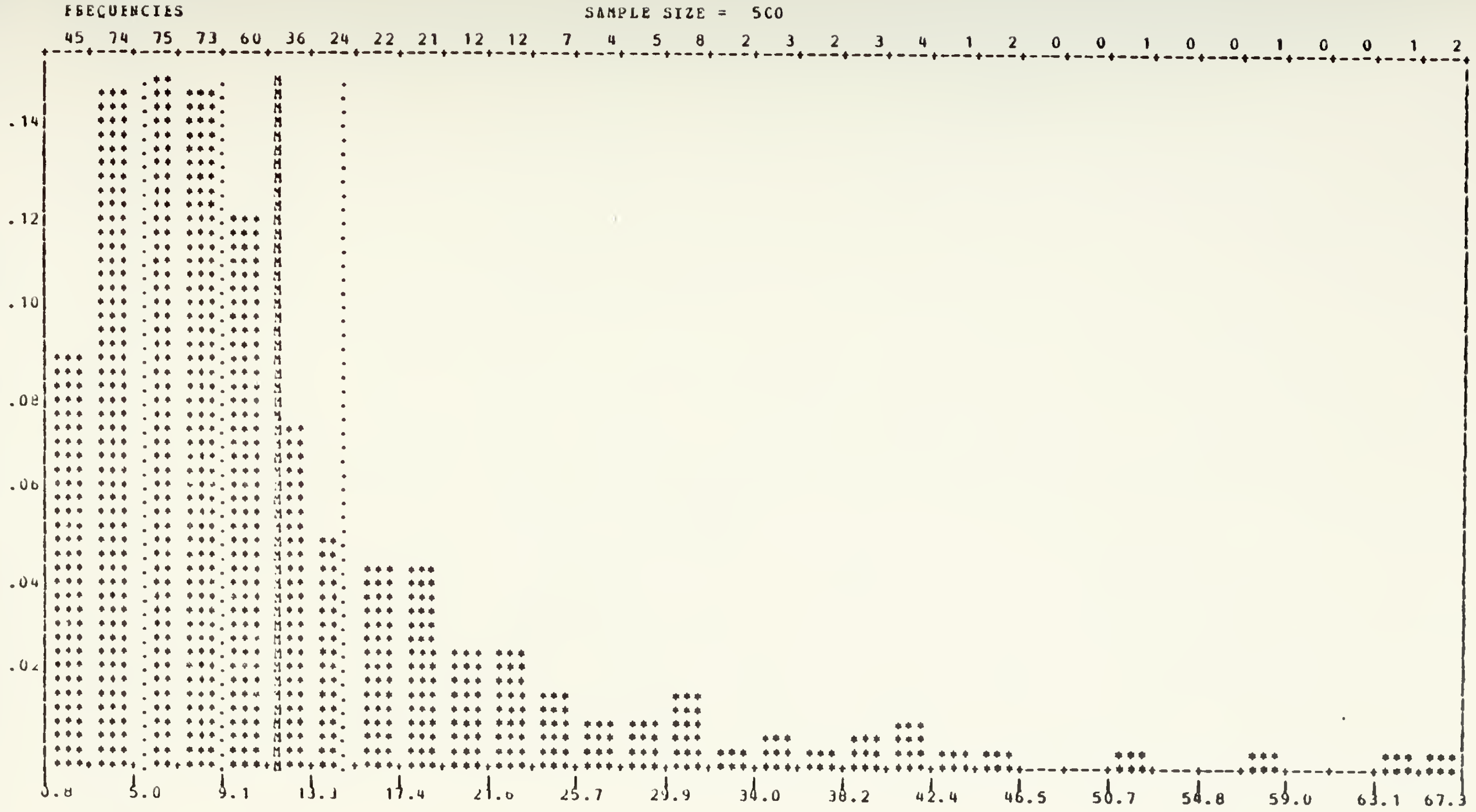
2. As for the steady state of the value of $E[W]$ and $E[\bar{W}]$ a large N was required for high t and/or r, also for the steady state of the distibution of W and $\bar{W}$ a large value of N was required. Informally testing their distributions, by using the Plotting Subroutines and the values of their parameters skewness and kurtosis, we may conclude that:

(i). The distribution of W, given that $W>0$, has an exponential form, if the plot results in a straight line under EXPLT and the γ_1 and γ_2 parameters have the values of about 2 and 6 respectively (see figures 23a-23d). This is true only if the steady state has been reached while it is not evident what their distribution is if the steady state has not been reached. Thus observing the figures 24a-24d, where the W_{10000} has not reached the steady state yet, we can obtain nothing about the distribution of W.

Note that for the M/M/1 queue the steady state distribution of W, given that $W>0$, is exponential. The distribution is not known for the correlated queue and this was one of the objectives of this study.

(ii). The distribution of $\bar{W}$ has the normal form if we get a straight line under NORMPL and if the values of skewness and kurtosis are around '0'. The normality assumption holds for the steady state only, while the non-steady state distribution looks like an exponentional. Figures 27a-27c give us the feeling that $\bar{W}_{10000}$ has an exponential form, since a straight line appears under EXPLT (see figure 27b) and the parameters γ_1, γ_2 have the values 2.3 and 7.3 respectively. These are close to the actual values 2 and 6 of the exponential distribution. We can see

from the plots that as the $\bar{W}$ gets close to the steady state, it leaves the exponentional form and comes closer and closer to the normal form. Figures 28a-28c, 23d show us that $\bar{W}_{10000}$ has already reached the steady state and furthermore its distribution has eventually taken the normal form, since a straight line appears under NORMPL (figure 28c) and the skewness and kurtosis have the value .7 and .8 respectively. Compare its distribution with the distribution of $\bar{W}_{10000}$ of figure 27b where $\bar{W}$ is not in the steady state. Again we may state here that convergence to the steady state of the distribution of W and $\bar{W}$ requires much larger N than convergence to the steady state of their mean values for the same t and r. And also the higher t and/or r the larger N should be. As an example we can see that even though the steady state (in value) of $\bar{W}$ for S#25#98 type has been reached for N=10000, the steady state in distribution has not been reached yet. See figures 24e-24g where we can observe that the distribution of $\bar{W}_{10000}$ starts to leave the exponential form and to go to the normal form.



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CENTRAL TENDENCY              SPREAD                           HIGHER CENTRAL MOMENTS    DISTRIBUTION

MEAN        1.144589E 01      VARIANCE    9.704788E 01         M3        2.227506E 03    MINIMUM                      8.059000E-01
MEDIAN      8.577194E 00      STD DEV     9.851289E 00         M4        9.727681E 04    .10 QUANTILE                 3.017599E 00
TRIMEAN     9.173805E 00      COEF VAR    8.606631E-01         SKEWNESS  2.329913E 00    .25 QUANTILE  (HINGE)        5.090400E 00
MIDMEAN     6.670585E 00      MEAN DEV    6.364244E 00         KURTOSIS  7.326501E 00    .50 QUANTILE  (MEDIAN)       8.577194E 00
MIDRANGE    3.404968E 01      RANGE       6.648758E 01         BETA1     2.214161E 03    .75 QUANTILE  (HINGE)        1.445045E 01
GEOM MEAN   8.509552E 00      MIDSPREAD   9.360047E 00         BETA2     9.661131E 04    .90 QUANTILE                 2.289980E 01
HARM MEAN   6.289999E 00                                                                 MAXIMUM                      6.729349E 01
```
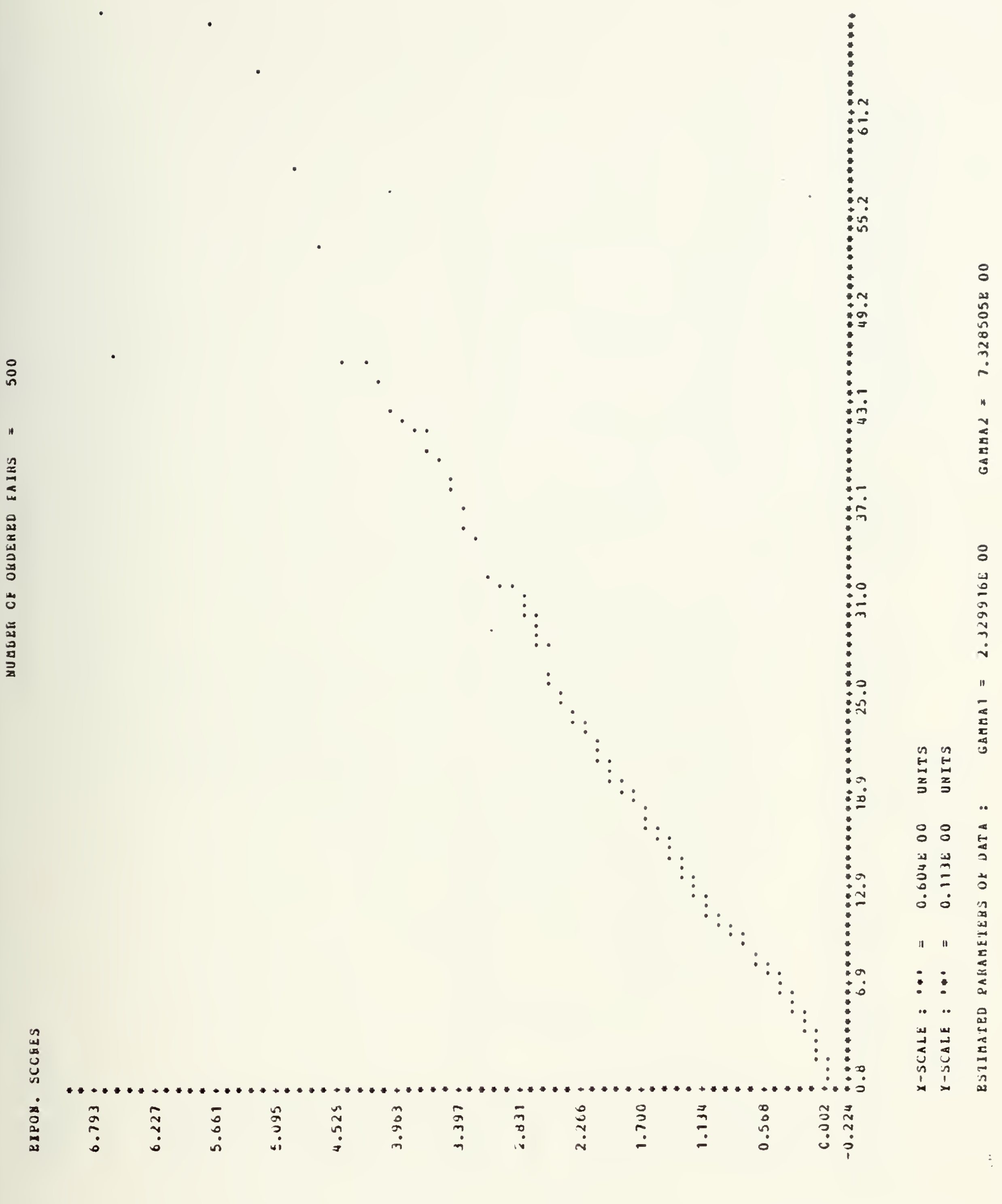

Figure 27a - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. HISTOGRAM OF THE CUMULATED AND AVERAGED WAITING TIMES $\overline{W}_{2000}$ FROM THE RUN S#50#90; m=500 REPLICATIONS, RX=.2; RS=.4 .



X-SCALE : ** = 0.604E 00 UNITS

Y-SCALE : ** = 0.113E 00 UNITS

ESTIMATED PARAMETERS OF DATA : GAMMA1 = 2.329916E 00 GAMMA2 = 7.328505E 00

Figure 27b - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. EXPONENTIAL PLOT (EXPLT) OF THE CUMULATED AND AVERAGED WAITING TIMES $\bar{W}_{2000}$ FROM THE RUN S#50#90; m=500 REPLICATIONS, RX=.2; RS=.4 .

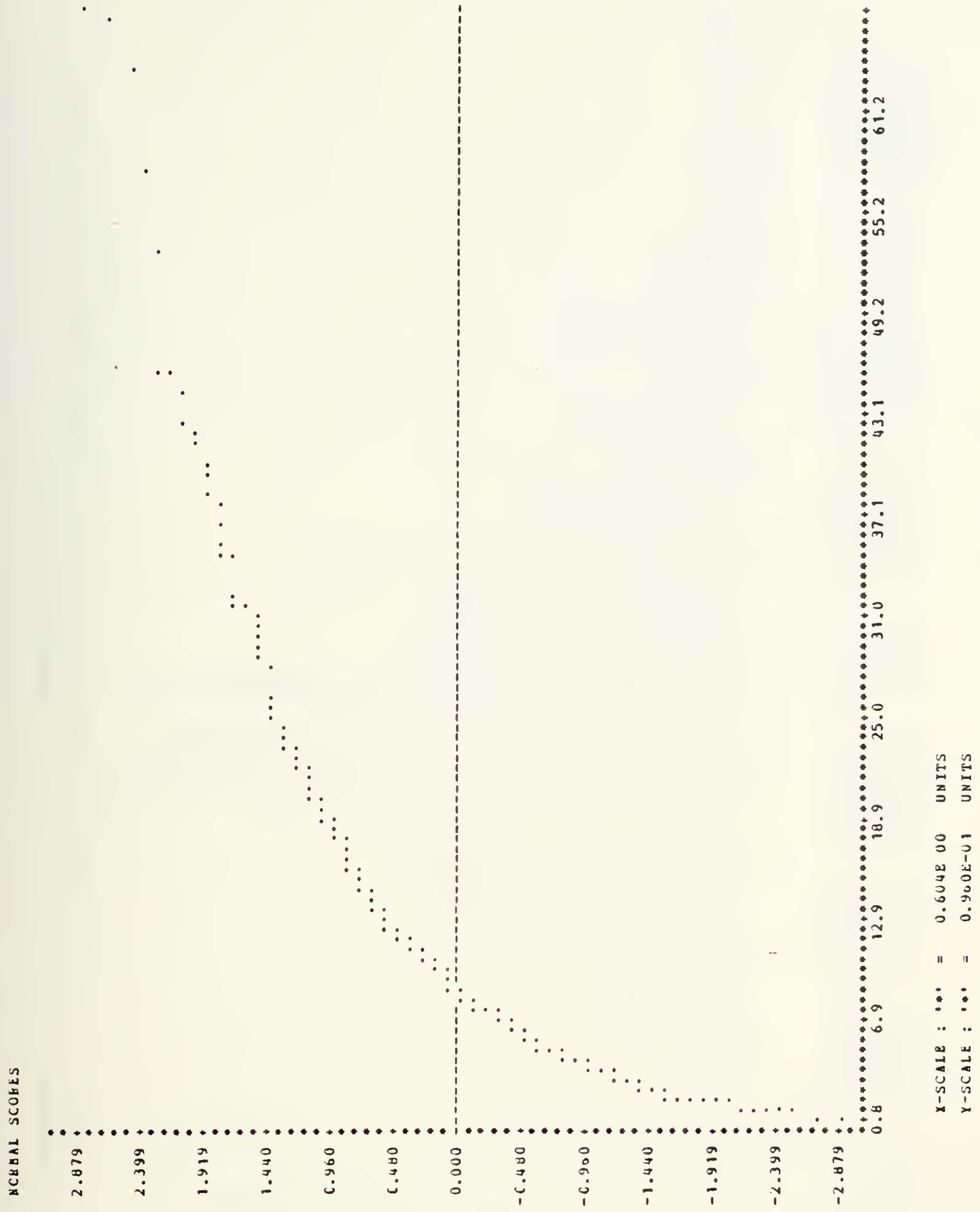
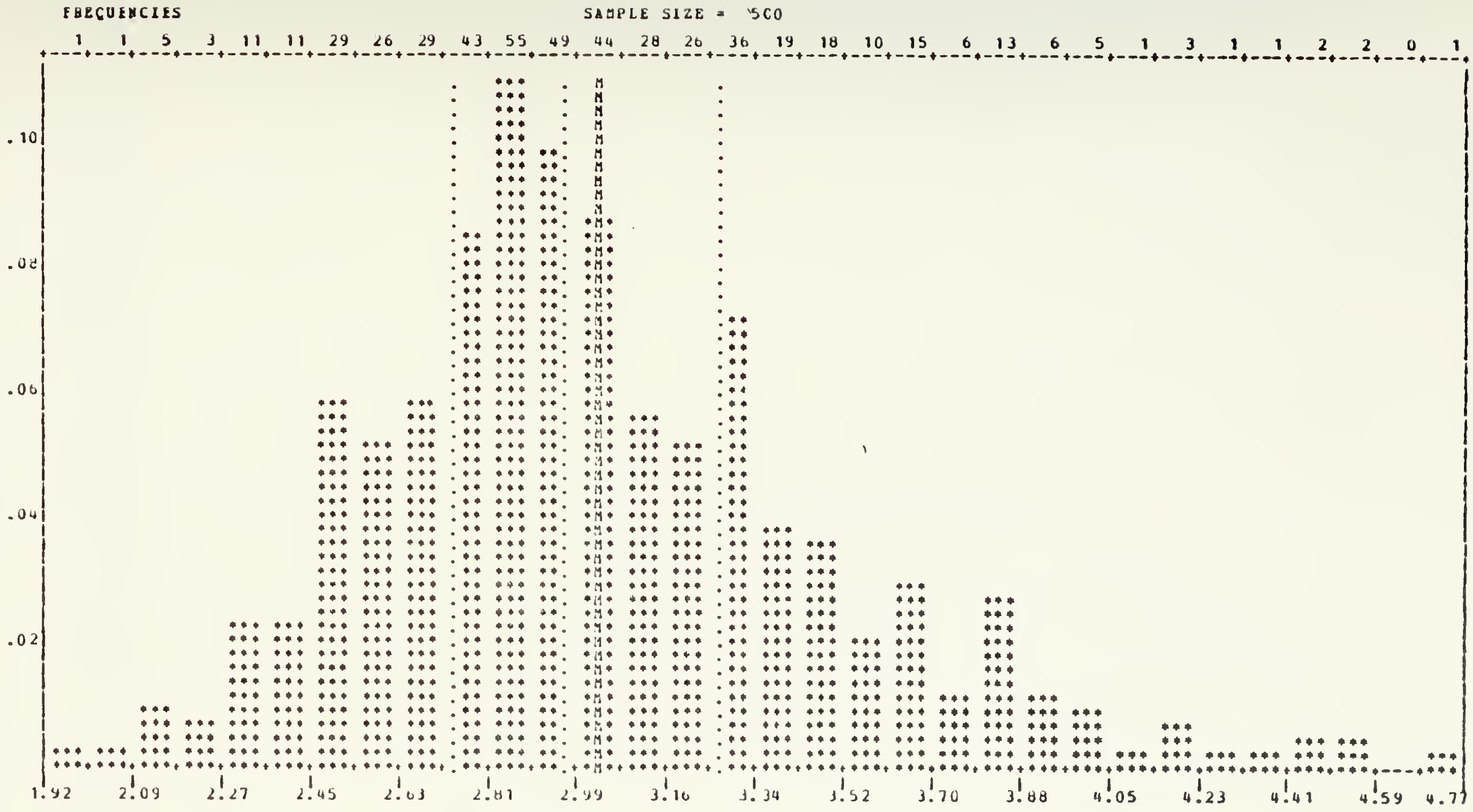


Figure 27c - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. NORMAL PLOT (NORMPL) OF THE CUMULATED AND AVERAGED WAITING TIMES $\overline{W}_{2000}$ FROM THE RUN S#50#90; m=500 REPLICATIONS, RX=.2; RS=.4 .



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CENTRAL TENDENCY              SPREAD                       HIGHER CENTRAL MOMENTS     DISTRIBUTION

MEAN        3.026470E 00      VARIANCE     1.985783E-01    M3         6.369668E-02    MINIMUM                           1.916499E 00
MEDIAN      2.951149E 00      STD DEV      4.456213E-01    M4         1.495504E-01    .10 QUANTILE                      2.501499E 00
TRIMEAN     2.977761E 00      COEF VAR     1.472412E-01    SKEWNESS   7.198124E-01    .25 QUANTILE  (HINGE)             2.726649E 00
MIDMEAN     2.977502E 00      MEAN DEV     3.413630E-01    KURTOSIS   7.924871E-01    .50 QUANTILE  (MEDIAN)            2.951149E 00
MIDRANGE    3.341999E 00      RANGE        2.851001E 00    BETA1      6.331503E-02    .75 QUANTILE  (HINGE)             3.282100E 00
GEOM MEAN   2.995045E 00      MIDSPREAD    5.554504E-01    BETA2      1.488258E-01    .90 QUANTILE                      3.640200E 00
HARM MEAN   2.964685E 00                                                              MAXIMUM                           4.767500E 00
```

Figure 28a - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. HISTOGRAM OF THE CUMULATED AND AVERAGED WAITING TIMES $\bar{W}_{2000}$ FROM THE RUN S#50#25; m=500 REPLICATIONS, RX=.2; RS=.4 .

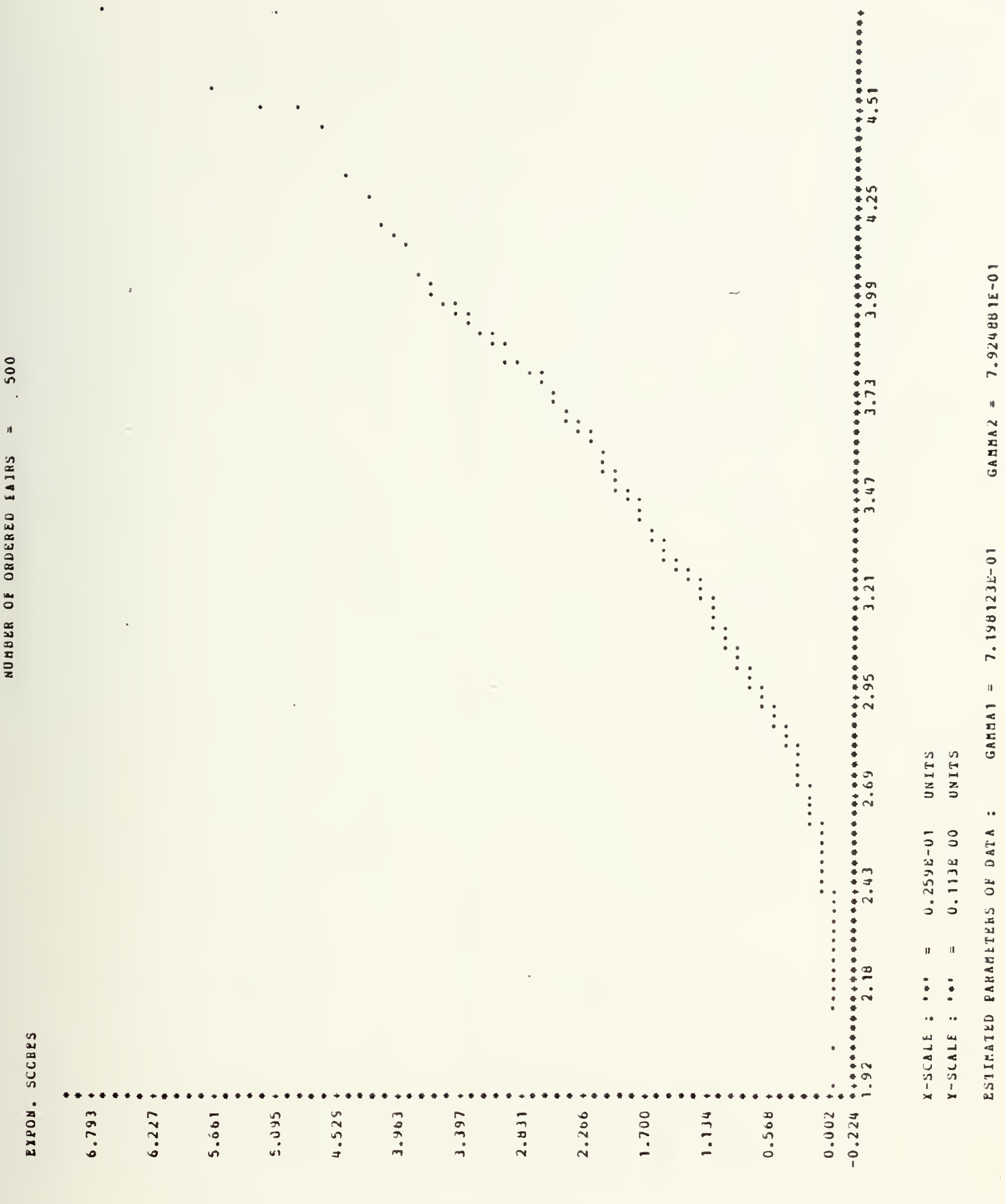


Figure 28b - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. EXPONENTIAL PLOT (EXPLT) OF THE CUMULATED AND AVERAGED WAITING TIMES $\bar{W}_{2000}$ FROM THE RUN S*50*25; m=500 REPLICATIONS, RX=.2; RS=.4 .

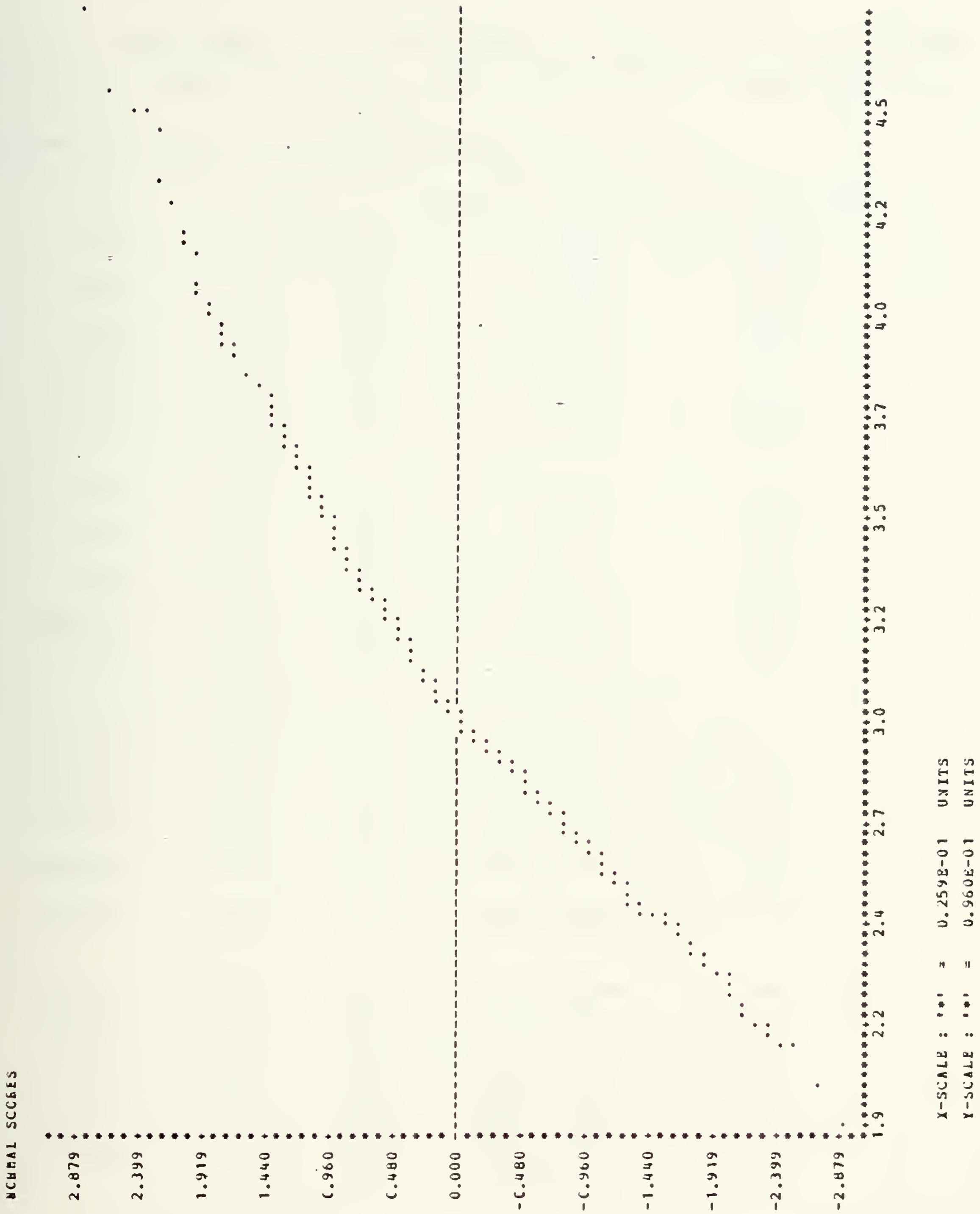


Figure 28c - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. NORMAL PLOT (NORMPL) OF THE CUMULATED AND AVERAGED WAITING TIMES $\bar{W}_{2000}$ FROM THE RUN S#50#25; m=500 REPLICATIONS, RX=.2; RS=.4 .

Continuing the analysis of the results we present below the results we got from all runs, concerning the values of $E[\overline{W}]$ for various values of t, r, N.

N	t	r	$E[\overline{W}]$(correlated)	$E[\overline{W}]$(M/M/1)
2000	.25	.25	.3720	.3332
2000	.25	.50	.4326	.3332
2000	.25	.90	.7578	.3332
10000	.25	.95	1.0576	.3332
10000	.25	.98	1.7077	.3332
2000	.50	.25	1.2106	.9985
2000	.50	.50	1.5811	.9985
2000	.50	.90	5.0785	.9985
10000	.50	.95	9.3196	.9985
10000	.50	.98	21.0542*	.9985
10000	.95	.25	23.5651	18.3528
10000	.95	.50	33.3367	18.3528
10000	.95	.90	123.9350*	18.3528
10000	.95	.95	196.2460*	18.3528
10000	.95	.98	342.9240*	18.3528
10000	.99	.25	62.3285*	53.3371
10000	.99	.50	77.9370*	53.3371
10000	.99	.90	193.9113*	53.3371
10000	.99	.95	271.7713*	53.3371
10000	.99	.98	426.6750*	53.3371

Note that these values have been scaled from the values obtained in the simulations by multiplying by RS. Thus for t=0.50 and r=0.25 the table gives $E[\overline{W}]$(correlated) as 1.2106; the value in Figure 23d is 3.026x0.4=1.2106 . The values marked * have not converged.

A study on these results gives us the following observations:

1. For given traffic intensity t the value of $E[\bar{W}]$ is not constant, as it is for M/M/1 queues, but it depends on the value of the correlation parameter r. Thus we see that the value of $E[\bar{W}]$ increase, as the value of r becomes larger and larger. Furthermore from figure 29a, where $E[\bar{W}]$ is plotted versus r we can see that the rate of increment is not constant (therefore is not a linear function of r) but it increases with r and eventually goes up infinitely as r goes to 1.

2. For given correlation r, we see that the value of $E[\bar{W}]$ increases with t, the same as happens in the M/M/1 queue (see figure 29b). But regardless of the value of r, $E[\bar{W}]$ for M/M/1 queue is always less than for the EARMA model. Thus we may state that the waiting time of the autocorrelated service time model is greater than the corresponding waiting time of the M/M/1 model.

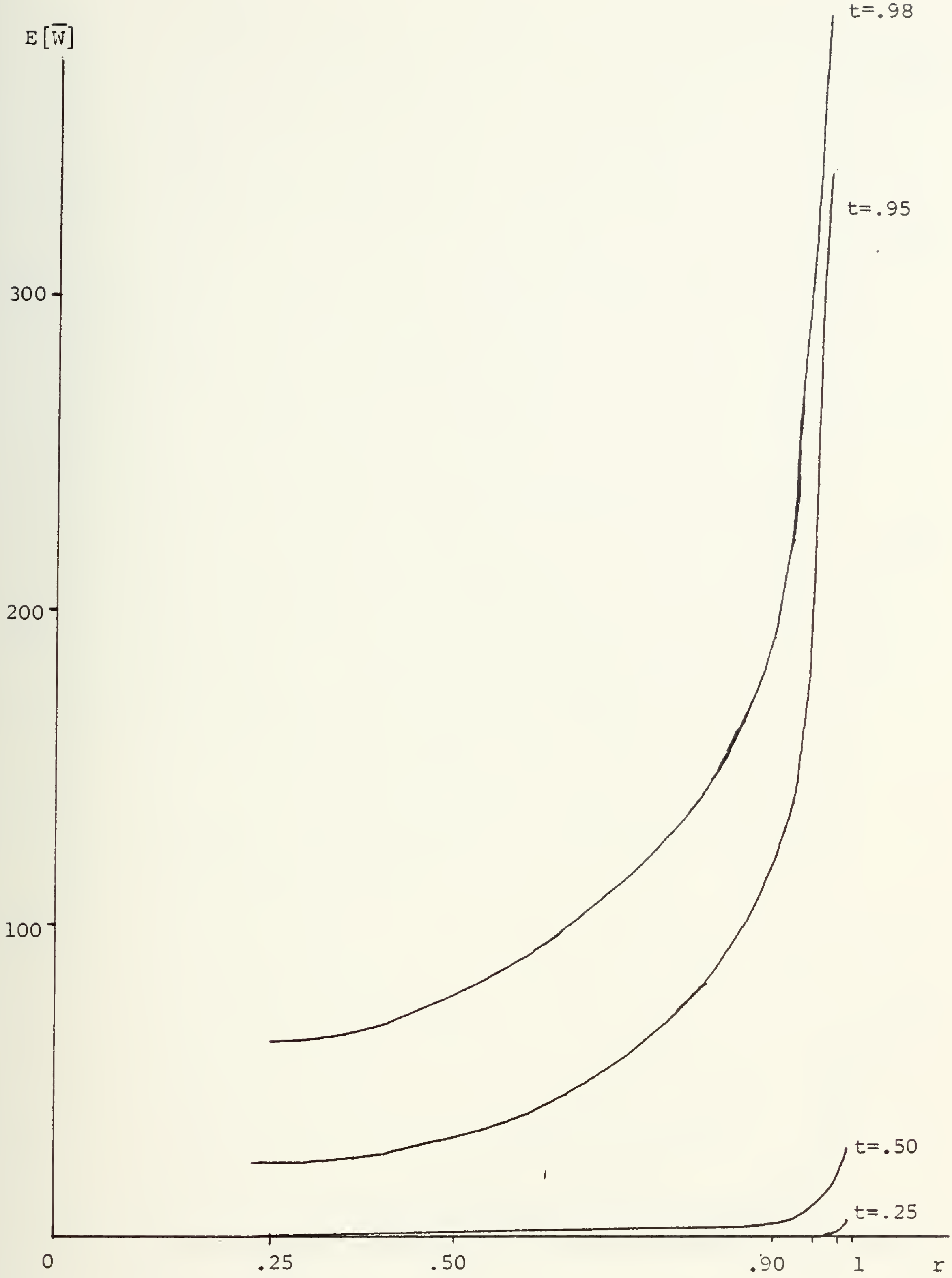


Figure 29a - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. PLOT OF E[$\overline{W}$] VERSUS r.

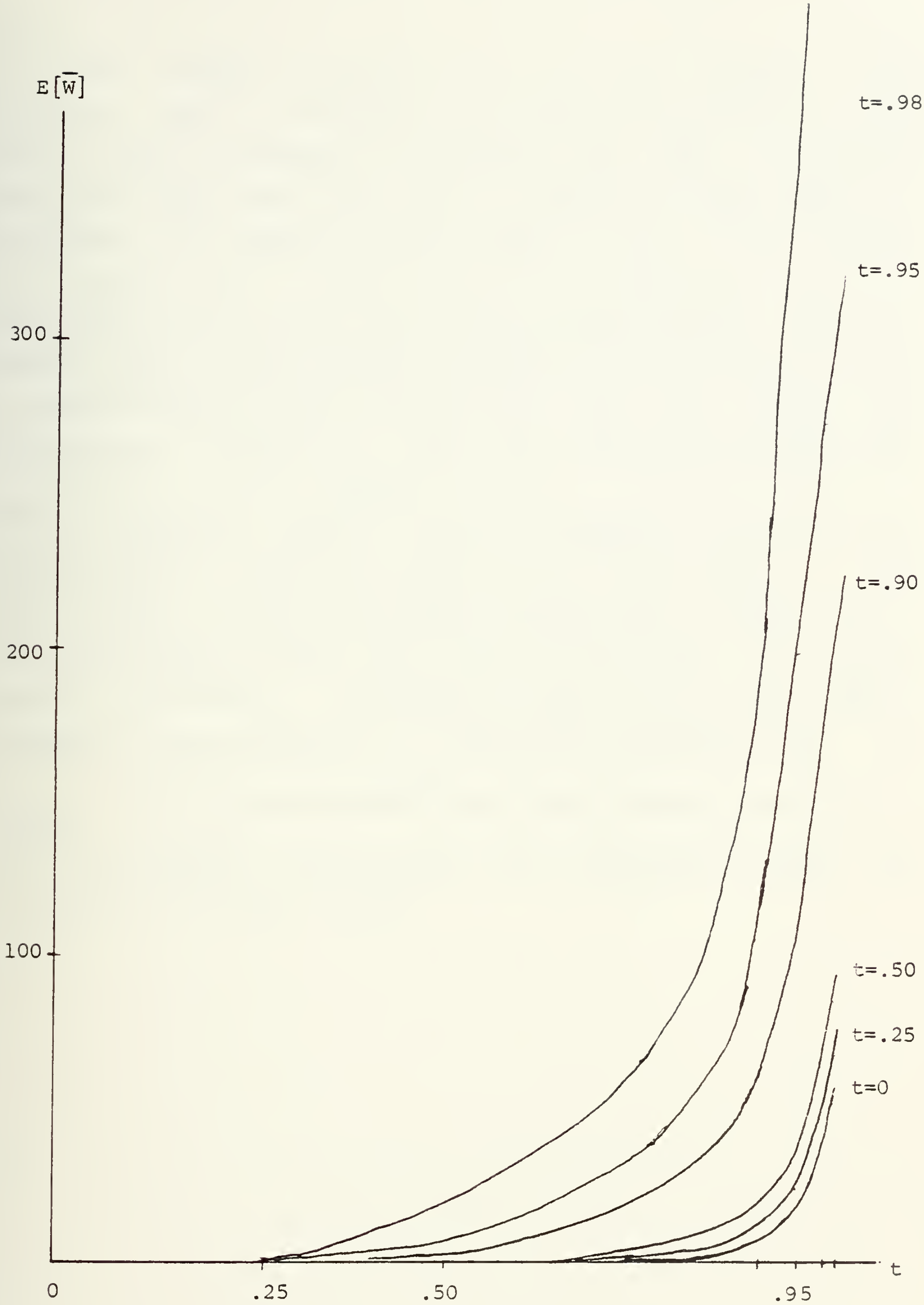


Figure 29b - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. PLOT OF E[$\overline{W}$] VERSUS t.

3. The values of $E[\overline{W}]$ for the type S#99#98 model increases with N following the curve in figure 30 a fact that is known to happen in the M/M/1 queue also. We can see from that figure that the rate of increment comes down as N increases and hopefully should come to zero as $\overline{W}$ reaches the steady state. Comparing then the M/M/1 queue with the correlated queue we can see that the M/M/1 queue converges faster than the correlated especially when r is high. Thus we can see from figures 32a-32c that both the value and distribution of the $\overline{W}_{10000}$ for M/M/1 queue have reached the steady state, but the corresponding $\overline{W}_{10000}$ of the correlated queue (figures 33a-d) has not (at least in its distribution). (See the skewness and kurtosis values for W1 BAR, W2 BAR, W3 BAR and W4 BAR in Figure 33d. These appear to be decreasing to zero, but have certainly not reached zero by N=10,000.) Looking at figure 31d, we can obtain the result that neither $\overline{W}_{10000}$ for the correlated queue, or $\overline{W}_{10000}$ for the uncorrelated queue has reached equilibrium. This is because the skewness and kurtosis values are definitely not close to zero.

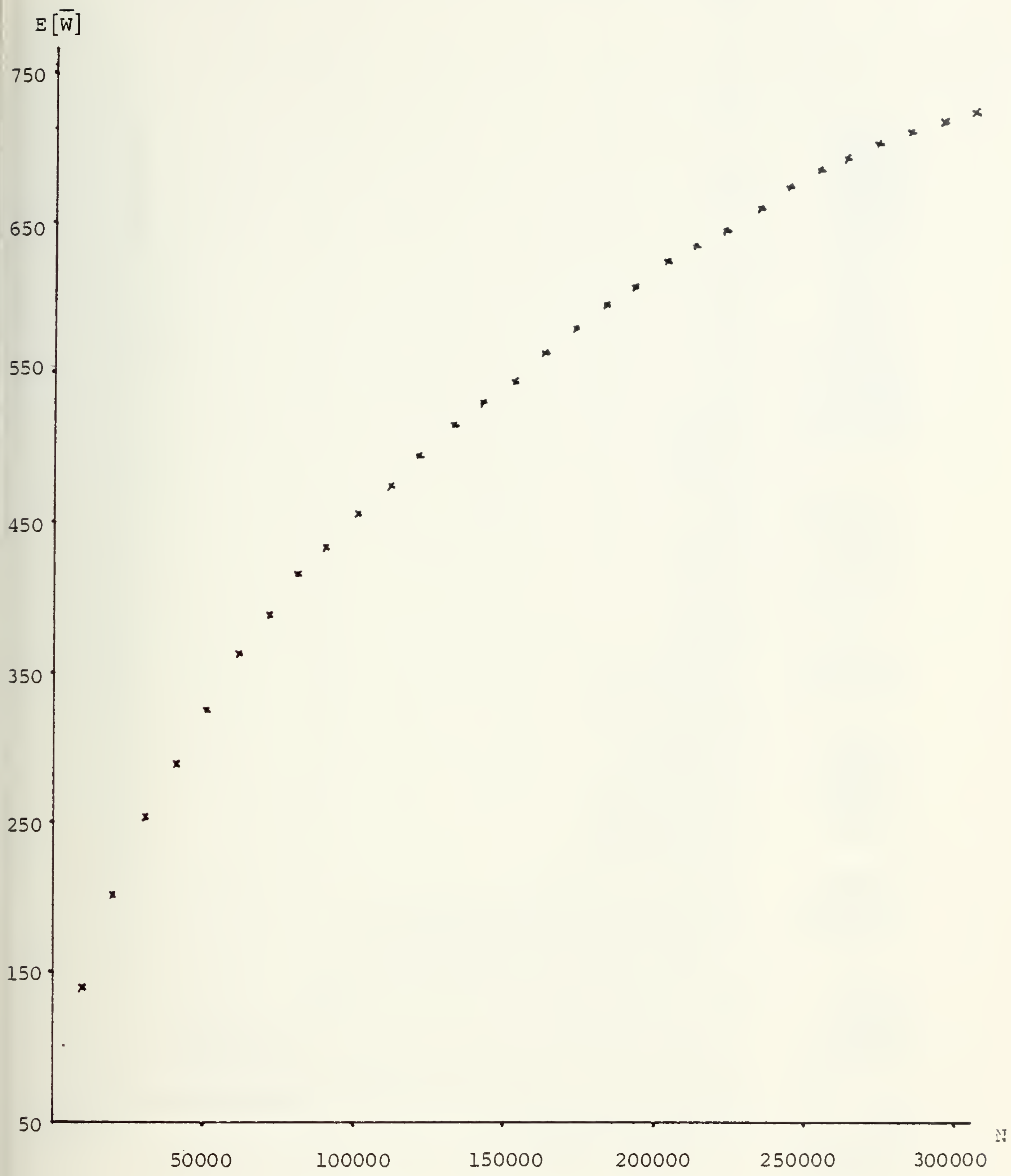
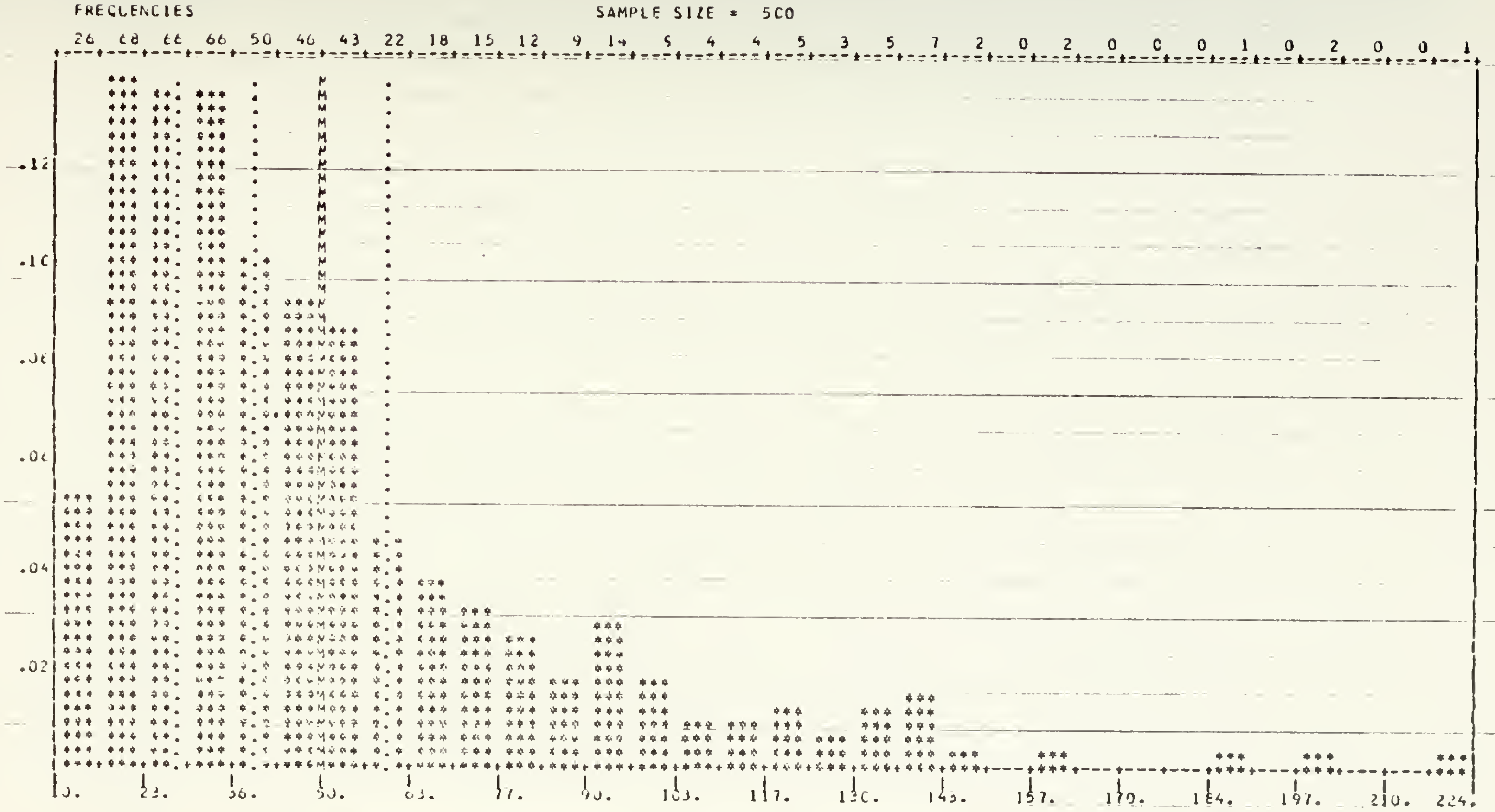


Figure 30 - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. PLOT OF E[$\overline{W}$] FROM THE RUN S#99#98, vs N.



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CENTRAL TENDENCY                 SPREAD                          HIGHER CENTRAL MOMENTS       DISTRIBUTION

MEAN        4.906149E 01         VARIANCE    1.086870E 03        M3          6.520440E 04     MINIMUM                    9.729099E 00
MEDIAN      3.925339E 01         STD DEV     3.296771E 01        M4          8.282023E 06     .10 QUANTILE               1.654109E 01
TRIMEAN     4.076234E 01         COFF VAR    6.719671E-01        SKEWNESS    1.815744E 00     .25 QUANTILE (HINGE)       2.633139E 01
MIDMEAN     4.364227E 01         MEAN DEV    2.272746E 01        KURTOSIS    4.096675E 00     .50 QUANTILE (MEDIAN)      3.925339E 01
MIDRANGE    1.167158E 02         RANGE       2.139735E 02        BETA1       6.481370E 04     .75 QUANTILE (HINGE)       5.621124E 01
GEOM MEAN   4.077974E 01         MIDSPREAD   3.187985E 01        BETA2       9.329059E 06     .90 QUANTILE               9.329465E 01
HARM MEAN   3.446984E 01                                                                      MAXIMUM                    2.237026E 02
```

Figure 31a - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. HISTOGRAM OF THE CUMULATED AND AVERAGED WAITING TIMES $\overline{W}_{10000}$ FROM THE RUN S#95#95; m=500 REPLICATIONS, RX=3.8; RS=4 .

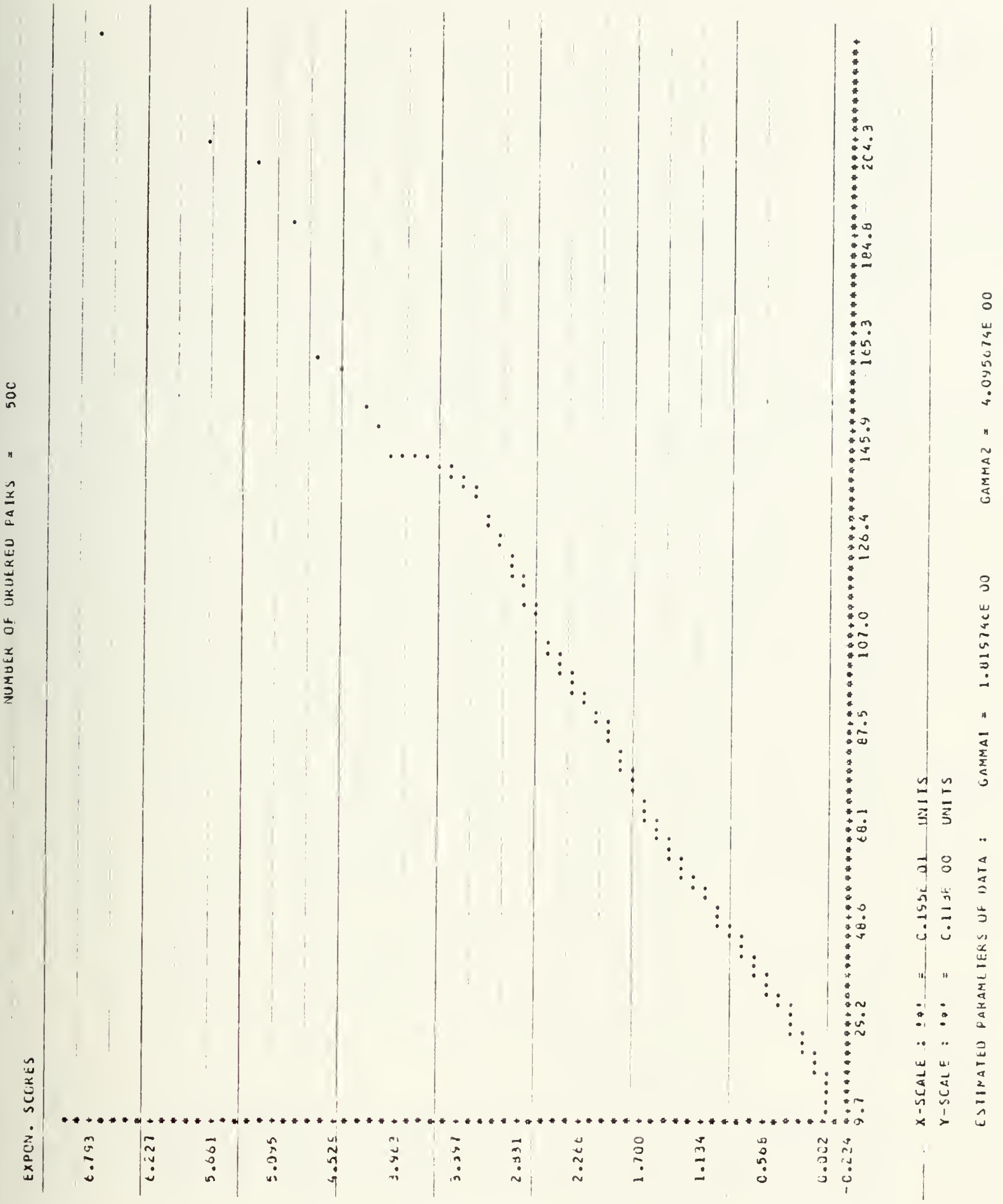


Figure 31b - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. EXPONENTIAL PLOT (EXPLT) OF THE CUMULATED AND AVERAGED WAITING TIMES $\overline{W}_{10000}$ FROM THE RUN S#95#95; m=500 REPLICATIONS, RX=3.8; RS=4 .

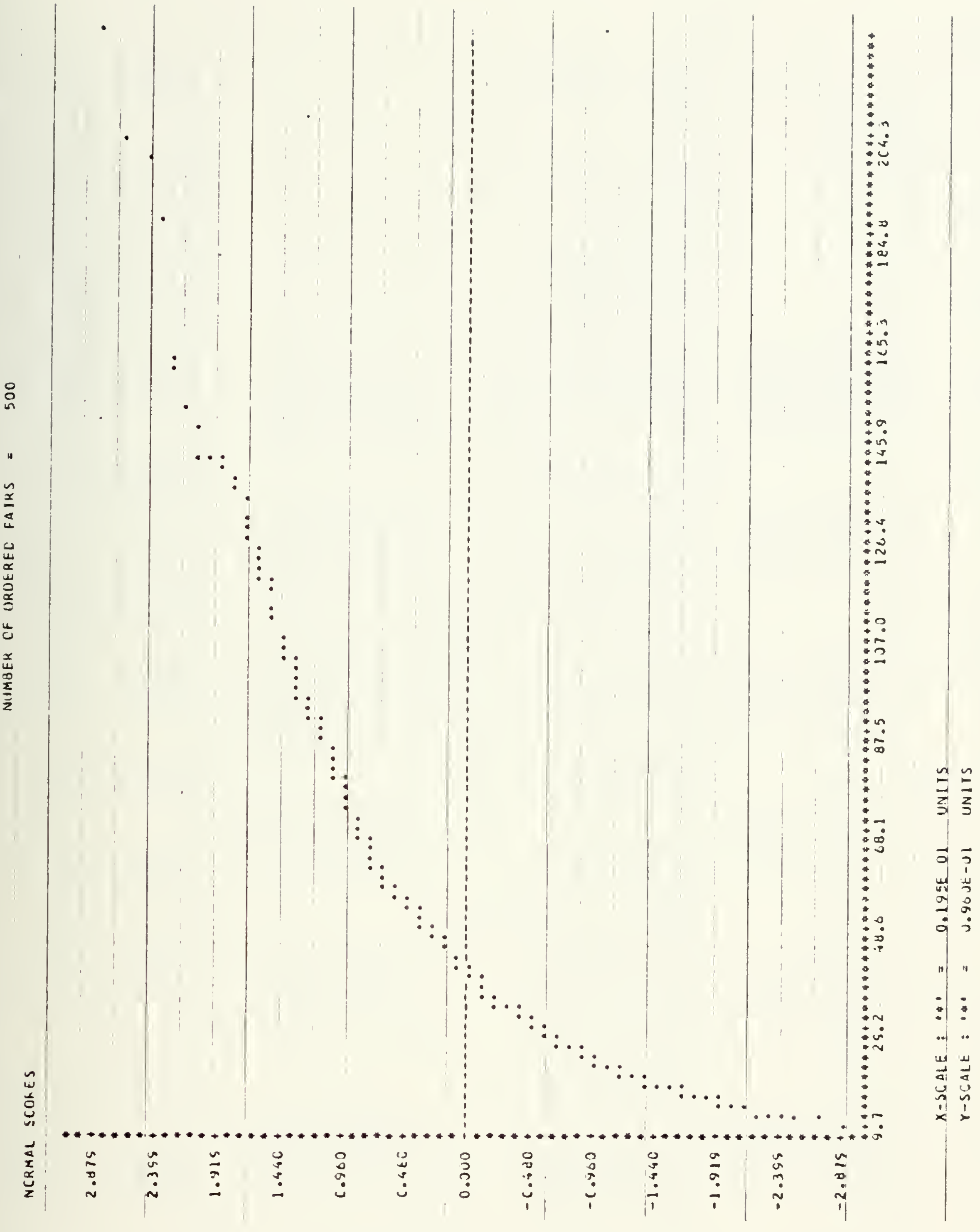


Figure 31c - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. NORMAL PLOT (NORMPL) OF THE CUMULATED AND AVERAGED WAITING TIMES $\overline{W}_{10000}$ FROM THE RUN S#95#95; m=500 REPLICATIONS, RX=3.8; RS=4 .

CORRELATED QUEUE

PARMTR	X 2	X 4	S 2	S 4	W 1 + OS	W 1	W 2 + OS	W 2	W 3 + OS	W 3	W 4 + OS	W 4
MEAN	1315.313	2630.721	1237.182	2486.526	41.83807	45.27930	51.31895	55.90302	60.93990	65.10672	66.76263	71.77728
S(MEAN)	0.8147	1.1435	4.8361	6.6572	1.89843	1.97078	2.32853	2.42396	2.70984	2.79321	3.05678	3.11920
ST. DEV.	18.2168	25.5697	108.1386	148.8606	42.45010	42.36031	52.06749	51.93159	60.59399	60.42641	68.35172	68.26694
SKEWNESS	-0.0260	-0.2054	0.0080	0.0779	1.34962	1.30511	1.35181	1.34899	1.50149	1.48071	1.53079	1.51625
KURTOS.	0.1782	0.1021	-0.1010	-0.1893	1.61421	1.49580	1.63750	1.45824	2.60343	2.54736	2.86478	2.82892
SMPL SIZE	500.0000	500.0000	500.0000	500.0000	500.00000	462.00000	500.00000	459.00000	500.00000	468.00000	500.00000	479.00000

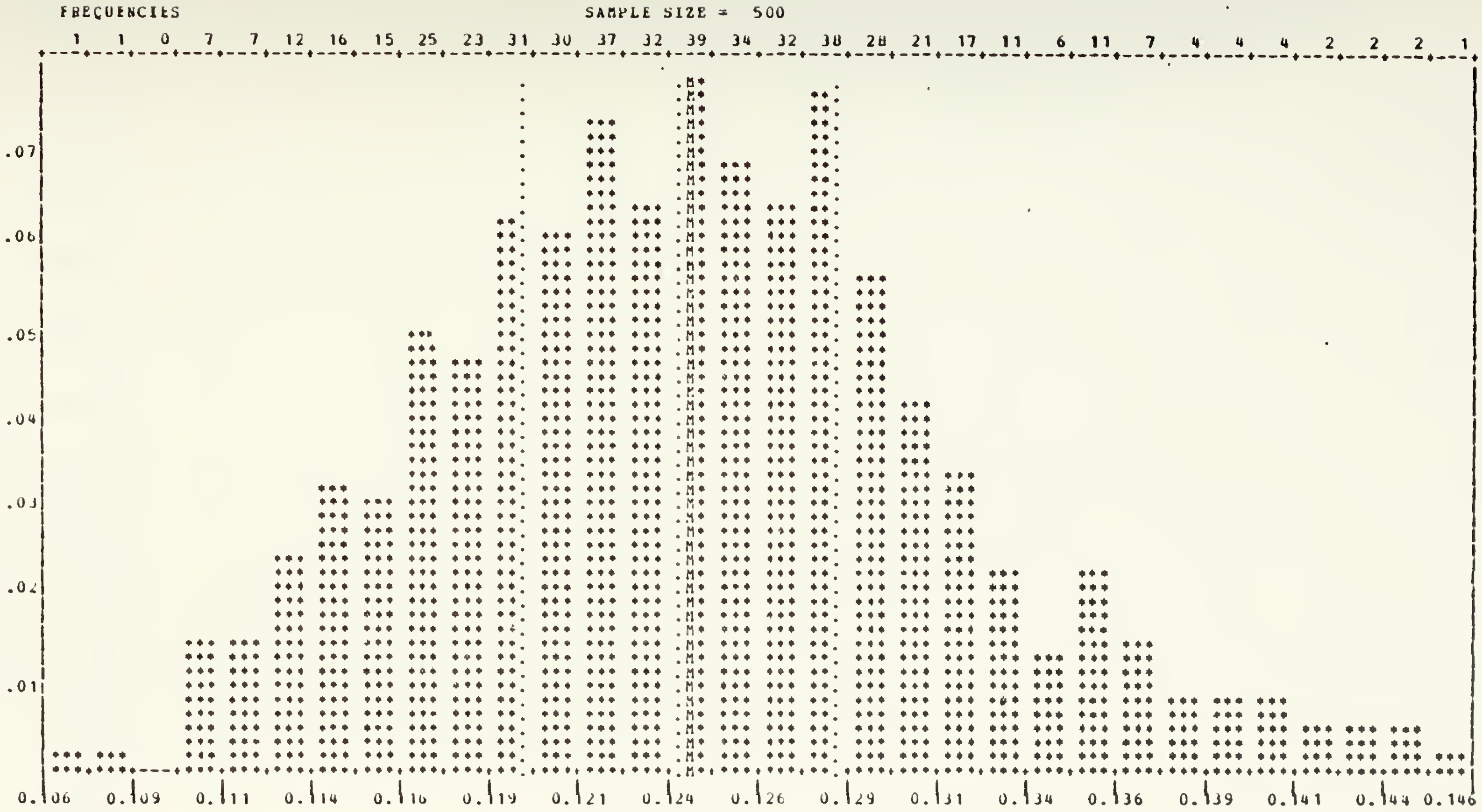
PARMTR	W 1 BAR	W 2 BAR	W 3 BAR	W 4 BAR	D 2	D 4
MEAN	23.4075623	37.4663427	44.2106934	49.0614925	-0.0024681	-0.0012617
S(MEAN)	0.9953651	1.1918144	1.3660173	1.4743595	0.0009737	0.0006720
ST. DEV.	22.2570456	26.6497555	30.5451050	32.9677124	0.0217730	0.0150274
SKEWNESS	1.5593653	1.4049540	1.6803305	1.8197441	0.0474807	0.0951524
KURTOS.	2.4249315	1.5788355	3.2507801	4.0956755	-0.1309185	-0.1001110
SMPL SIZE	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000

UNCORRELATED QUEUE

PARMTR	X 2	X 4	S 2	S 4	W 1 + OS	W 1	W 2 + OS	W 2	W 3 + OS	W 3	W 4 + OS	W 4
MEAN	1315.313	2630.721	1248.937	2493.015	4.67354	4.89889	4.62055	4.80307	4.68507	5.03770	4.81275	5.00286
S(MEAN)	0.8147	1.1435	0.7893	1.1556	0.21404	0.21914	0.22754	0.23264	0.22734	0.23652	0.24120	0.24675
ST. DEV.	18.2168	25.5697	17.6482	25.8401	4.78606	4.78612	5.08790	5.10225	5.08358	5.10018	5.39336	5.41172
SKEWNESS	-0.0260	-0.2054	0.2370	0.0945	1.70479	1.69067	2.40052	2.35665	2.00690	1.98799	2.61904	2.61761
KURTOS.	0.1782	0.1021	0.0275	0.0293	3.06231	2.98011	8.73606	8.68852	5.76017	5.66502	11.39535	11.36556
SMPL SIZE	500.0000	500.0000	500.0000	500.0000	500.00000	477.00000	500.00000	481.00000	500.00000	465.00000	500.00000	481.00000

PARMTR	W 1 BAR	W 2 BAR	W 3 BAR	W 4 BAR	D 2	D 4
MEAN	4.0176134	4.3716478	4.5287819	4.5882139	-0.0001173	-0.0001126
S(MEAN)	0.0939731	0.0992808	0.0945966	0.0903672	0.0002257	0.0001636
ST. DEV.	2.1013041	2.2199869	2.1152468	2.0206738	0.0050478	0.0036572
SKEWNESS	1.9176502	2.6313343	3.0940571	3.5639057	0.2856022	0.2776073
KURTOS.	4.5888758	10.0241823	16.1784058	22.3170471	-0.1193161	0.1654472
SMPL SIZE	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000

Figure 31d - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. TABULATION OF SAMPLE STATISTICS FOR THE DISTRIBUTIONS OF CUMULATED INTERARRIVAL TIMES AT N=5000 AND N=10000 (X 2 AND X 4), CUMULATED SERVICE TIMES (S 2 AND S 4), WAITING TIMES, WITH OR WITHOUT ZEROS (WI + 0, WI, I FOR CASE N=2500, 5000, 7500, 10000), CUMULATED WAITING TIMES (W1 BAR etc.) AND THE AVERAGED DIFFERENCES BETWEEN S2 AND X2 (D2); OBTAINED FROM m=500 REPLICATIONS OF THE RUN S*95*95; RX=2³•8, RS=4 .



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CENTRAL TENDENCY                 SPREAD                           HIGHER CENTRAL MOMENTS          DISTRIBUTION

MEAN        1.244181E-01         VARIANCE   4.012340E-05          M3          1.012634E-07        MINIMUM                         1.062000E-01
MEDIAN      1.241000E-01         STD DEV    6.937101E-03          M4          6.975842E-09        .10 QUANTILE                    1.154000E-01
TRIMEAN     1.241124E-01         COEF VAR   5.575635E-02          SKEWNESS    3.033320E-01        .25 QUANTILE  (HINGE)           1.196000E-01
MIDMEAN     1.242405E-01         MEAN DEV   5.510590E-03          KURTOSIS    1.392841E-02        .50 QUANTILE  (MEDIAN)          1.241000E-01
MIDRANGE    1.263500E-01         RANGE      4.030001E-02          BETA1       1.006566E-07        .75 QUANTILE  (HINGE)           1.286500E-01
GEOM MEAN   1.242266E-01         MIDSPREAD  9.049952E-03          BETA2       6.951712E-09        .90 QUANTILE                    1.331000E-01
HARM MEAN   1.240363E-01                                                                          MAXIMUM                         1.465000E-01
```

Figure 32a - M/M/1 QUEUE. HISTOGRAM OF THE CUMULATED AND AVERAGED WAITING TIMES $\overline{W}_{10000}$ FROM THE RUN S#50#95; m=500 REPLICATIONS, RX=4; RS=8 .

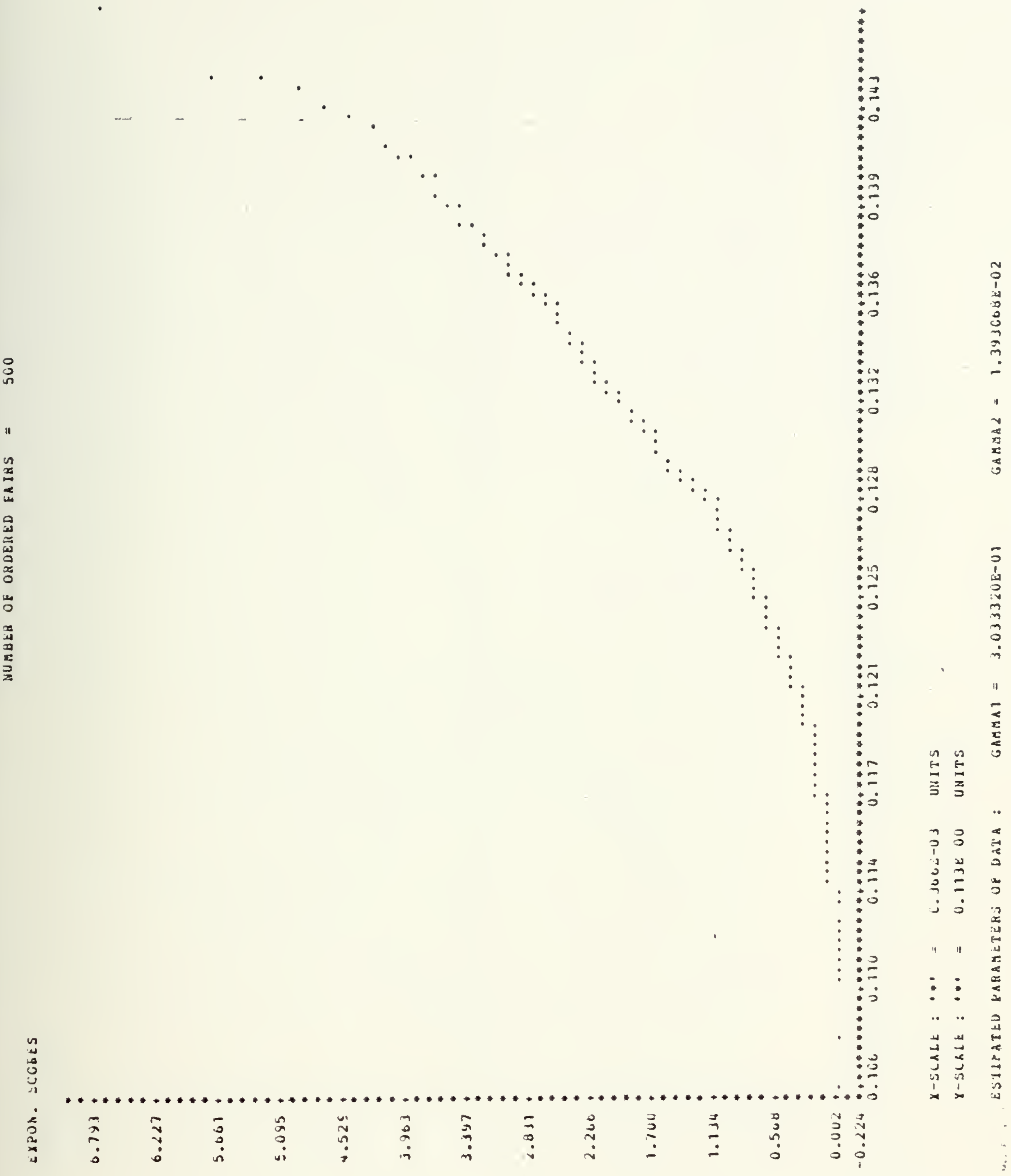


Figure 32b - M/M/1 QUEUE. EXPONENTIAL PLOT (EXPLT) OF THE CUMULATED AND AVERAGED WAITING TIMES W_{10000} FROM THE RUN S#50#95; m=500 REPLICATIONS, RX=4; RS=8 .

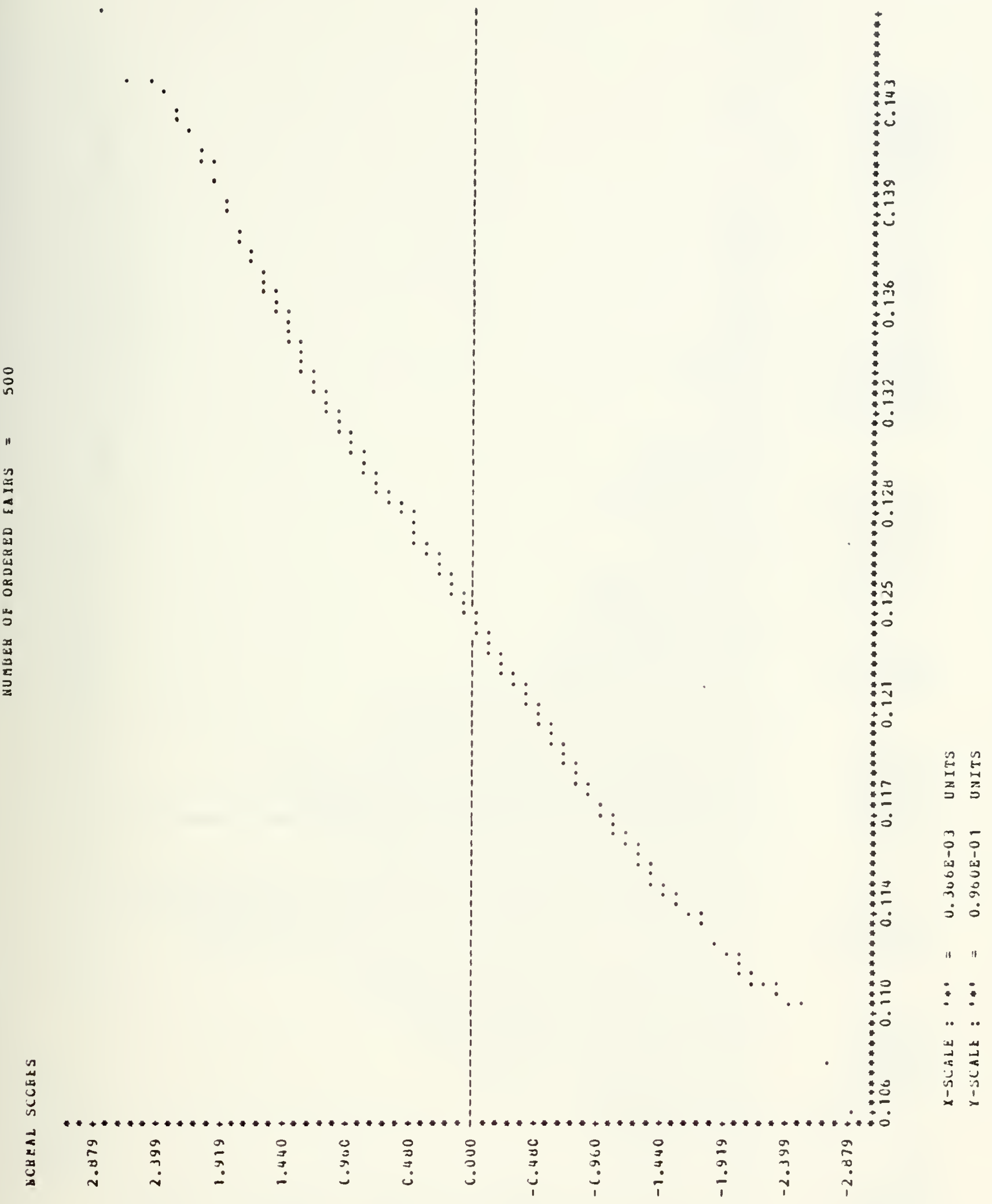
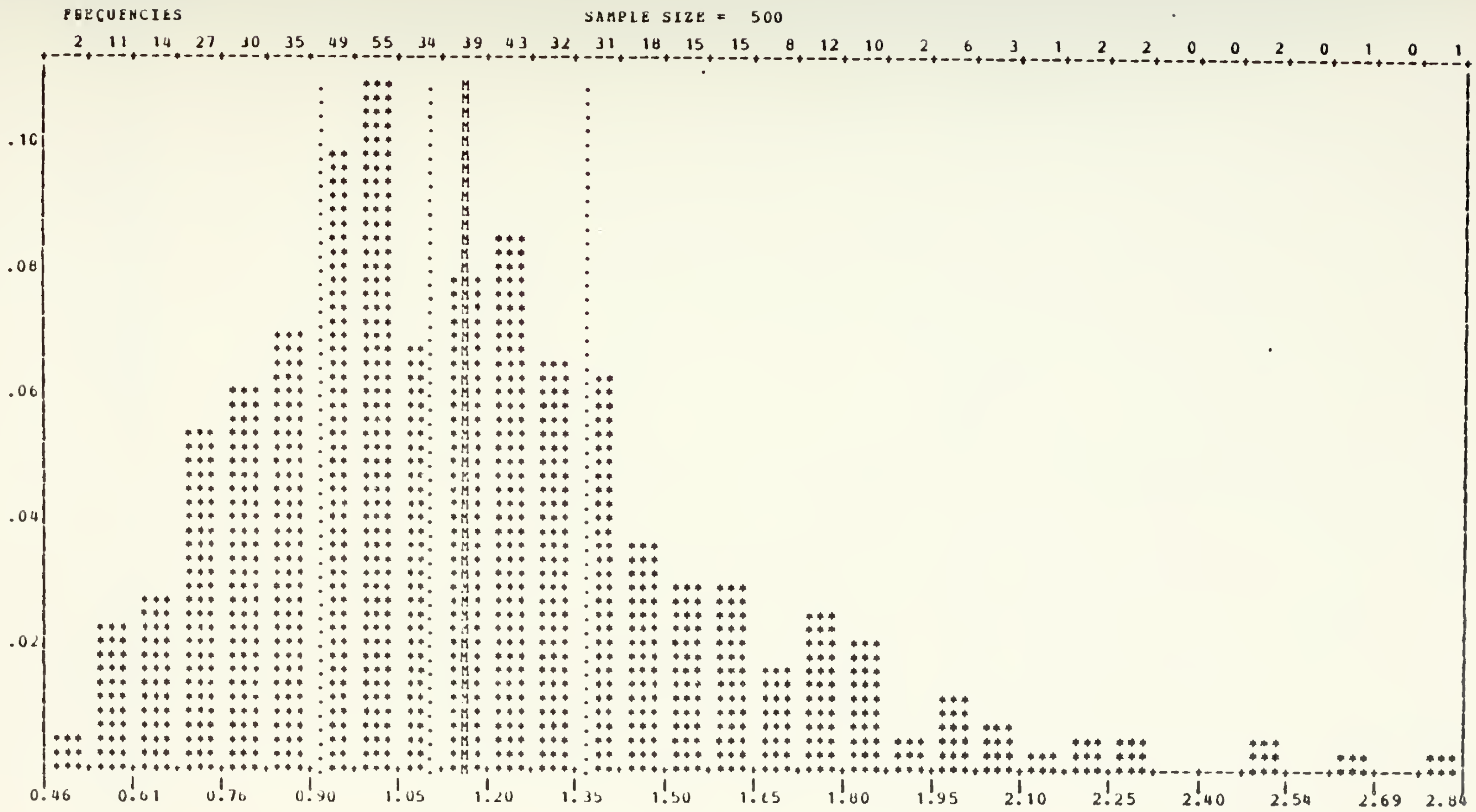


Figure 32c - M/M/1 QUEUE. NORMAL PLOT (NORMPL) OF THE CUMULATED AND AVERAGED WAITING TIMES $\bar{W}_{10000}$ FROM THE RUN S#50#95; m=500 REPLICATIONS, RX=4; RS=8 .



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CENTRAL TENDENCY              SPREAD                          HIGHER CENTRAL MOMENTS        DISTRIBUTION

MEAN         1.164949E 00     VARIANCE     1.347327E-01       M3         4.885203E-02       MINIMUM                          4.572999E-01
MEDIAN       1.105249E 00     STD DEV      3.670596E-01       M4         8.285785E-02       .10 QUANTILE                     7.496000E-01
TRIMEAN      1.121549E 00     COEF VAR     3.150862E-01       SKEWNESS   9.876090E-01       .25 QUANTILE  (HINGE)            9.147496E-01
MIDMEAN      1.118072E 00     MEAN DEV     2.807866E-01       KURTOSIS   1.564440E 00       .50 QUANTILE  (MEDIAN)           1.105249E 00
MIDRANGE     1.650149E 00     RANGE        2.385699E 00       BETA1      4.855932E-02       .75 QUANTILE  (HINGE)            1.360950E 00
GEOM MEAN    1.111442E 00     MIDSPREAD    4.461999E-01       BETA2      8.241218E-02       .90 QUANTILE                     1.634800E 00
HARM MEAN    1.060657E 00                                                                   MAXIMUM                          2.842999E 00
```

Figure 33a - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. HISTOGRAM OF THE CUMULATED AND AVERAGED WAITING TIMES W_{10000} FROM THE RUN S#50#95; m=500 REPLICATIONS, RX=4; RS=8 .

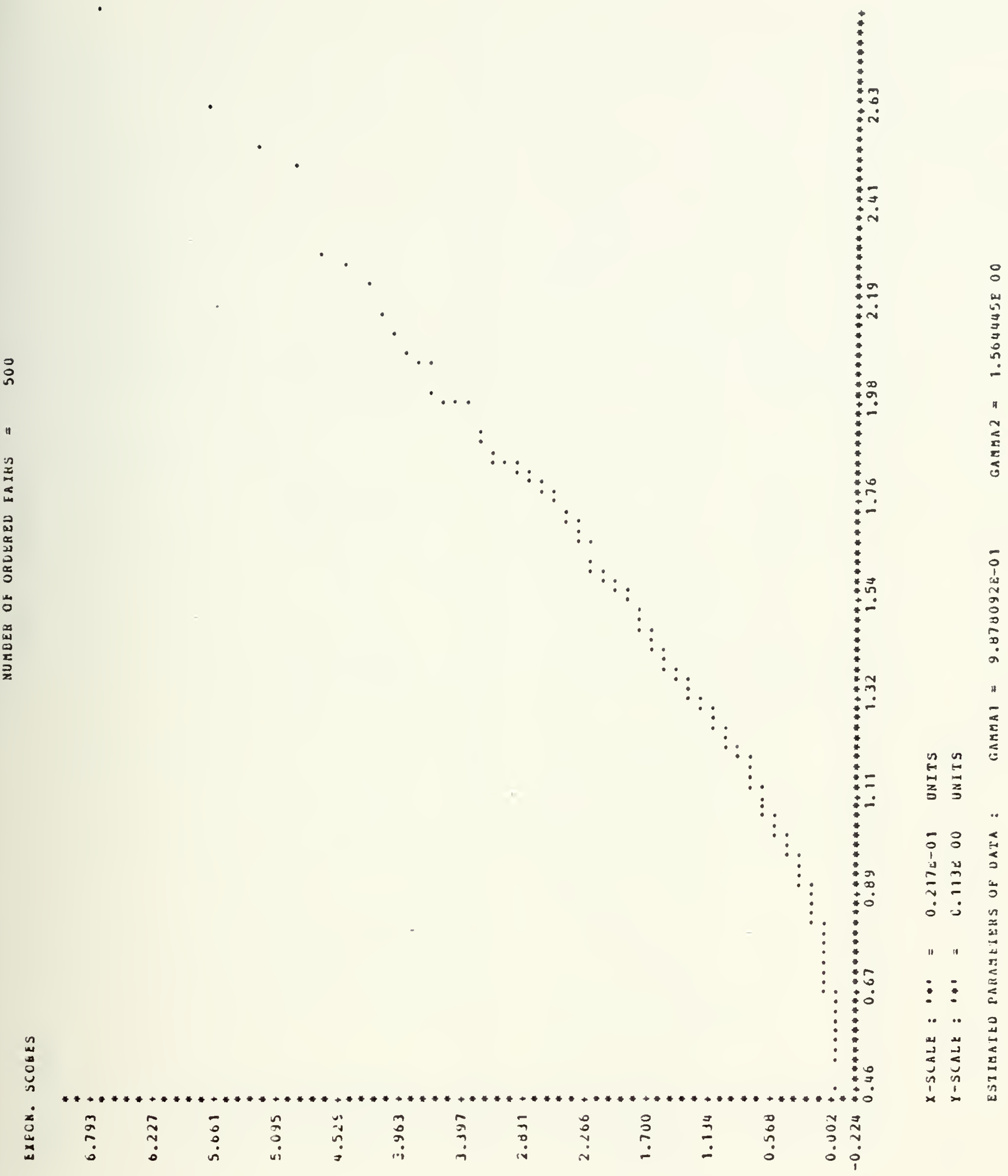


Figure 33b - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. EXPONENTIAL PLOT (EXPLT) OF THE CUMULATED AND AVERAGED WAITING TIMES $\overline{W}_{10000}$ FROM THE RUN S#50#95; m=500 REPLICATIONS, RX=4; RS=8 .

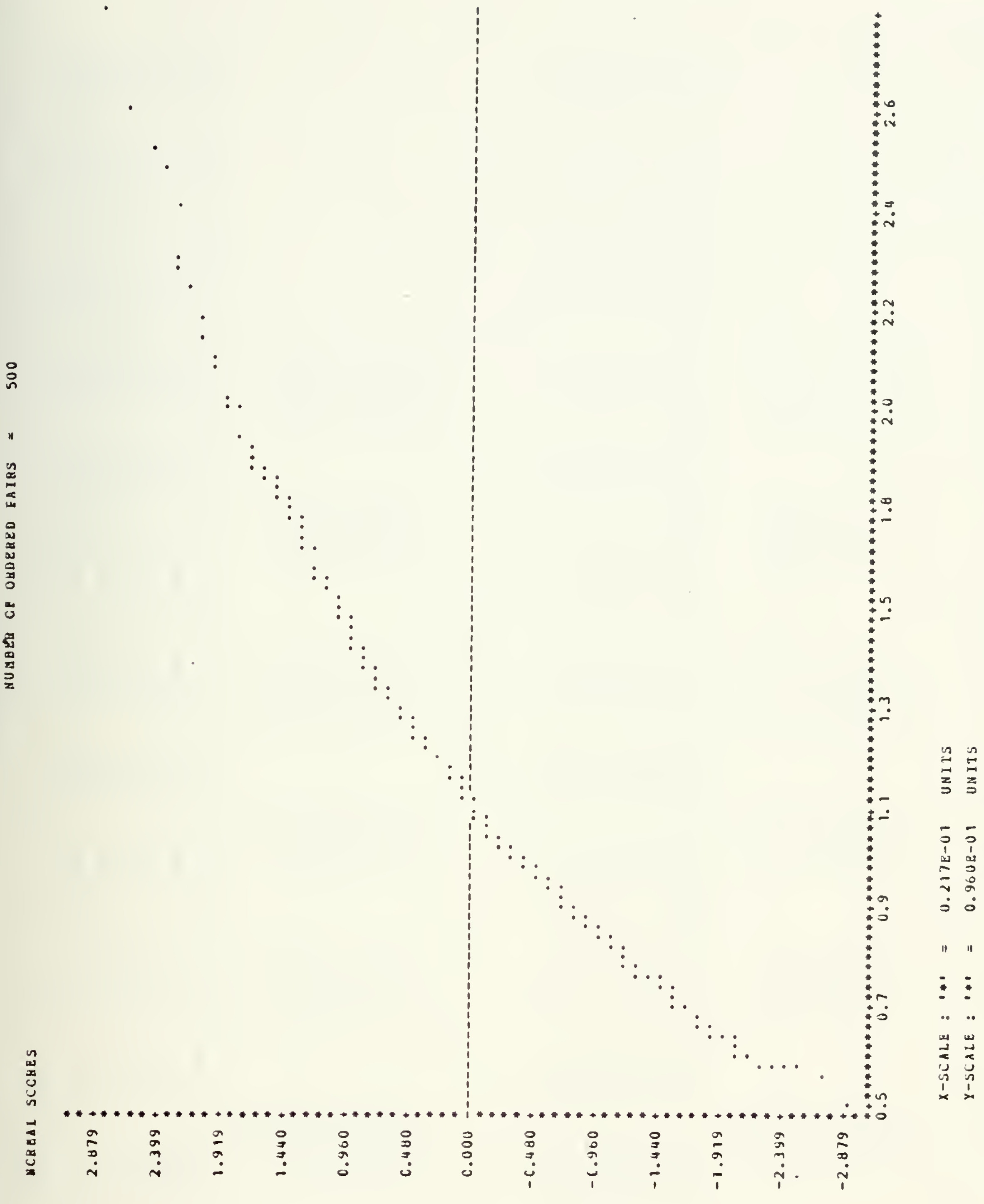


Figure 33c - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. NORMAL PLOT (NORMPL) OF THE CUMULATED AND AVERAGED WAITING TIMES $\bar{W}_{10000}$ FROM THE RUN S#50#95; m=500 REPLICATIONS, RX=4; RS=8 .

CORRELATED QUEUE

PARMTR	X 2	X 4	S 2	S 4	W 1 + OS	W 1	W 2 + OS	W 2	W 3 + OS	W 3	W 4 + OS	W 4
MEAN	1249.528	2499.137	618.464	1242.831	1.10977	2.19321	1.11538	2.24876	1.06352	2.42813	1.22590	2.26181
S(MEAN)	0.7739	1.0863	2.4174	3.3282	0.12115	0.21912	0.11521	0.20915	0.11936	0.24341	0.13227	0.22578
ST. DEV.	17.3060	24.2914	54.0547	74.4219	2.70909	3.48526	2.57611	3.29364	2.66889	3.60218	2.95759	3.71680
SKEWNESS	-0.0260	-0.2054	0.0081	0.0779	4.06847	2.85987	3.71708	2.56583	3.46160	2.07075	3.87350	2.78568
KURTOS.	0.1782	0.1019	-0.1009	-0.1893	22.30673	11.21000	17.27135	8.11302	12.78089	3.89474	19.32036	10.00603
SMPL SIZE	500.0000	500.0000	500.0000	500.0000	500.00000	253.00000	500.00000	248.00000	500.00000	219.00000	500.00000	271.00000

PARMTR	W 1 BAR	W 2 BAR	W 3 BAR	W 4 BAR	D 2	D 4
MEAN	1.1084175	1.1266356	1.1483183	1.1649494	-0.0012129	-0.0006308
S(MEAN)	0.0312151	0.0227753	0.0187032	0.0164154	0.0005011	0.0003470
ST. DEV.	0.6979922	0.5092710	0.4182159	0.3670596	0.0112045	0.0077584
SKEWNESS	2.1959677	1.4561481	1.0985785	0.9878090	0.0689812	0.0926246
KURTOS.	9.9568624	3.4234600	1.6901579	1.5644398	-0.1391373	-0.0192480
SMPL SIZE	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000

UNCORRELATED QUEUE

PARMTR	X 2	X 4	S 2	S 4	W 1 + OS	W 1	W 2 + OS	W 2	W 3 + OS	W 3	W 4 + OS	W 4
MEAN	1249.528	2499.137	624.340	1248.573	0.12396	0.23127	0.11235	0.22292	0.13562	0.28855	0.13832	0.25426
S(MEAN)	0.7739	1.0863	0.3945	0.5777	0.00913	0.01406	0.00817	0.01284	0.01183	0.02112	0.01018	0.01556
ST. DEV.	17.3060	24.2914	8.8215	12.9186	0.20414	0.23016	0.18258	0.20378	0.26447	0.32379	0.22771	0.25668
SKEWNESS	-0.0260	-0.2054	0.2370	0.0946	2.27932	1.60773	2.11332	1.39834	3.08997	2.15515	2.87303	2.36012
KURTOS.	0.1782	0.1019	0.0273	0.0296	5.69143	2.64637	4.49536	1.63658	12.53375	6.30021	13.30220	10.00946
SMPL SIZE	500.0000	500.0000	500.0000	500.0000	500.00000	268.00000	500.00000	252.00000	500.00000	235.00000	500.00000	272.00000

PARMTR	W 1 BAR	W 2 BAR	W 3 BAR	W 4 BAR	D 2	D 4
MEAN	0.1242058	0.1242967	0.1244361	0.1244181	-0.0000377	-0.0000564
S(MEAN)	0.0005801	0.0004132	0.0003521	0.0003102	0.0001731	0.0001237
ST. DEV.	0.0129725	0.0092394	0.0078728	0.0069371	0.0038699	0.0027650
SKEWNESS	0.4769654	0.3978707	0.1912445	0.3033320	0.1976776	0.2695435
KURTOS.	0.0190020	0.0134497	0.1096792	0.0139284	-0.0576553	-0.0083609
SMPL SIZE	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000

Figure 33d - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. TABULATION OF SAMPLE STATISTICS FOR THE DISTRIBUTIONS OF CUMULATED INTERARRIVAL TIMES AT N=5000 AND N=10000 (X 2 AND X 4), CUMULATED SERVICE TIMES (S 2 AND S 4), WAITING TIMES, WITH OR WITHOUT ZEROS (WI + 0, WI, I FOR CASE N=2500, 5000, 7500, 10000), CUMULATED WAITING TIMES (W1 BAR etc.) AND THE AVERAGED DIFFERENCES BETWEEN S2 AND X2 (D2); OBTAINED FROM m=500 REPLICATIONS OF THE RUN S#50#95; RX=4; RS=8 .

Another result obtained from the simulation concerns the comparison of convergence of W and $\overline{W}$. From the programs we have run we observe that $\overline{W}$ converges slightly faster than W (at least for low t and r). Also the standard deviation of $\overline{W}$ is much less than of W. In figure 24d we can see that $\overline{W}_{10000}$ has already reached the steady state but not W_{10000}. Compare also the values of their standard deviations (.005 versus .2). Figure 34 gives the results of the type S#50#98 model for N=10000. We can see that while $\overline{W}$ is already close to the steady state, the fact that W has not reached it yet forces us to continue the simulation by increasing the value of N.

CORRELATED QUEUE

PARMTR	X 2	X 4	S 2	S 4	W 1 + OS	W 1	W 2 + OS	W 2	W 3 + OS	W 3	W 4 + OS	W 4
MEAN	1249.528	2499.137	619.247	1242.193	2.45903	4.95772	2.51849	4.51893	2.15700	4.81473	3.53149	6.28379
S(MEAN)	0.7739	1.0863	3.7038	5.4699	0.30512	0.58225	0.30600	0.55818	0.27387	0.56325	0.36354	0.59776
ST. DEV.	17.3060	24.2914	82.8198	122.3099	6.91220	9.16986	6.84229	8.93067	6.12388	8.42990	8.12891	10.02027
SKEWNESS	-0.0260	-0.2054	0.1998	0.2401	5.75748	4.17755	4.30115	2.56666	4.78043	3.15608	3.26556	2.30045
KURTOS.	0.1782	0.1019	-0.0918	0.1234	50.79430	27.25491	22.45009	10.56665	27.63379	11.59648	12.12130	5.70293
SMPL SIZE	500.0000	500.0000	500.0000	500.0000	500.00000	246.00000	500.00000	256.00000	500.00000	224.00000	500.00000	281.00000

PARMTR	W 1 BAR	W 2 BAR	W 3 BAR	W 4 BAR	D 2	D 4
MEAN	2.6210012	2.5524268	2.6078873	2.6317711	-0.0010562	-0.0006943
S(MEAN)	0.1089657	0.0816494	0.0686976	0.0617550	0.0007576	0.0005500
ST. DEV.	2.4370852	1.8257370	1.5361271	1.3808632	0.0169393	0.0122991
SKEWNESS	2.1862211	2.1719894	1.9596882	1.5567585	0.1692343	0.2547509
KURTOS.	5.9681206	7.2361326	6.4361839	3.3528345	-0.1520901	0.1401863
SMPL SIZE	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000

UNCORRELATED QUEUE

PARMTR	X 2	X 4	S 2	S 4	W 1 + OS	W 1	W 2 + OS	W 2	W 3 + OS	W 3	W 4 + OS	W 4
MEAN	1249.528	2499.137	624.340	1248.573	0.12396	0.23127	0.11235	0.22292	0.13562	0.28855	0.13832	0.25426
S(MEAN)	0.7739	1.0863	0.3945	0.5777	0.00913	0.01406	0.00817	0.01284	0.01183	0.02112	0.01018	0.01556
ST. DEV.	17.3060	24.2914	8.8215	12.9186	0.20414	0.23016	0.18258	0.20378	0.26447	0.32379	0.22771	0.25668
SKEWNESS	-0.0260	-0.2054	0.2370	0.0946	2.27932	1.60773	2.11332	1.59834	3.06997	2.15515	2.87303	2.38012
KURTOS.	0.1782	0.1019	0.0273	0.0296	5.69143	2.64637	4.49536	1.63658	12.53375	6.30021	13.30220	10.00946
SMPL SIZE	500.0000	500.0000	500.0000	500.0000	500.00000	268.00000	500.00000	252.00000	500.00000	235.00000	500.00000	272.00000

PARMTR	W 1 BAR	W 2 BAR	W 3 BAR	W 4 BAR	D 2	D 4
MEAN	0.1242058	0.1242967	0.1244361	0.1244181	-0.0000377	-0.0000564
S(MEAN)	0.0005801	0.0004132	0.0003521	0.0003102	0.0001731	0.0001237
ST. DEV.	0.0129725	0.0092394	0.0078728	0.0069371	0.0038699	0.0027650
SKEWNESS	0.4769654	0.3978707	0.1912445	0.3033320	0.1976776	0.2695435
KURTOS.	0.0150020	0.0134497	0.1096792	0.0139264	-0.0576553	-0.0083609
SMPL SIZE	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000

Figure 34 - QUEUE WITH EAR1 AUTOREGRESSIVE SERVICE TIMES AND POISSON INPUT. TABULATION OF SAMPLE STATISTICS FOR THE DISTRIBUTIONS OF CUMULATED INTERARRIVAL TIMES AT N=5000 AND N=10000 (X 2 AND X 4), CUMULATED SERVICE TIMES (S 2 AND S 4), WAITING TIMES, WITH OR WITHOUT ZEROS (WI + 0, WI, I FOR CASE N=2500, 5000, 7500, 10000), CUMULATED WAITING TIMES (W1 BAR etc.) AND THE AVERAGED DIFFERENCES BETWEEN S2 AND X2 (D2); OBTAINED FROM m=500 REPLICATIONS OF THE RUN S#5C#98; RX=4; RS=8 .

2. Queue with Cross-Correlated Service Time

The model for the cross-correlated queue is:

$\{X_i\} = \mathcal{E}_i$

$\{S_i\}$ is EARMA(1,1) over $\{ E_i, {}_i RX/RS, {}_{i-1}RX/RS, \ldots \}$

That is:

$$S_i = \begin{cases} \beta E_i & \text{w.p. } \beta \\ \beta E_i + A_i & \text{w.p } (1-\beta) \end{cases}$$

where

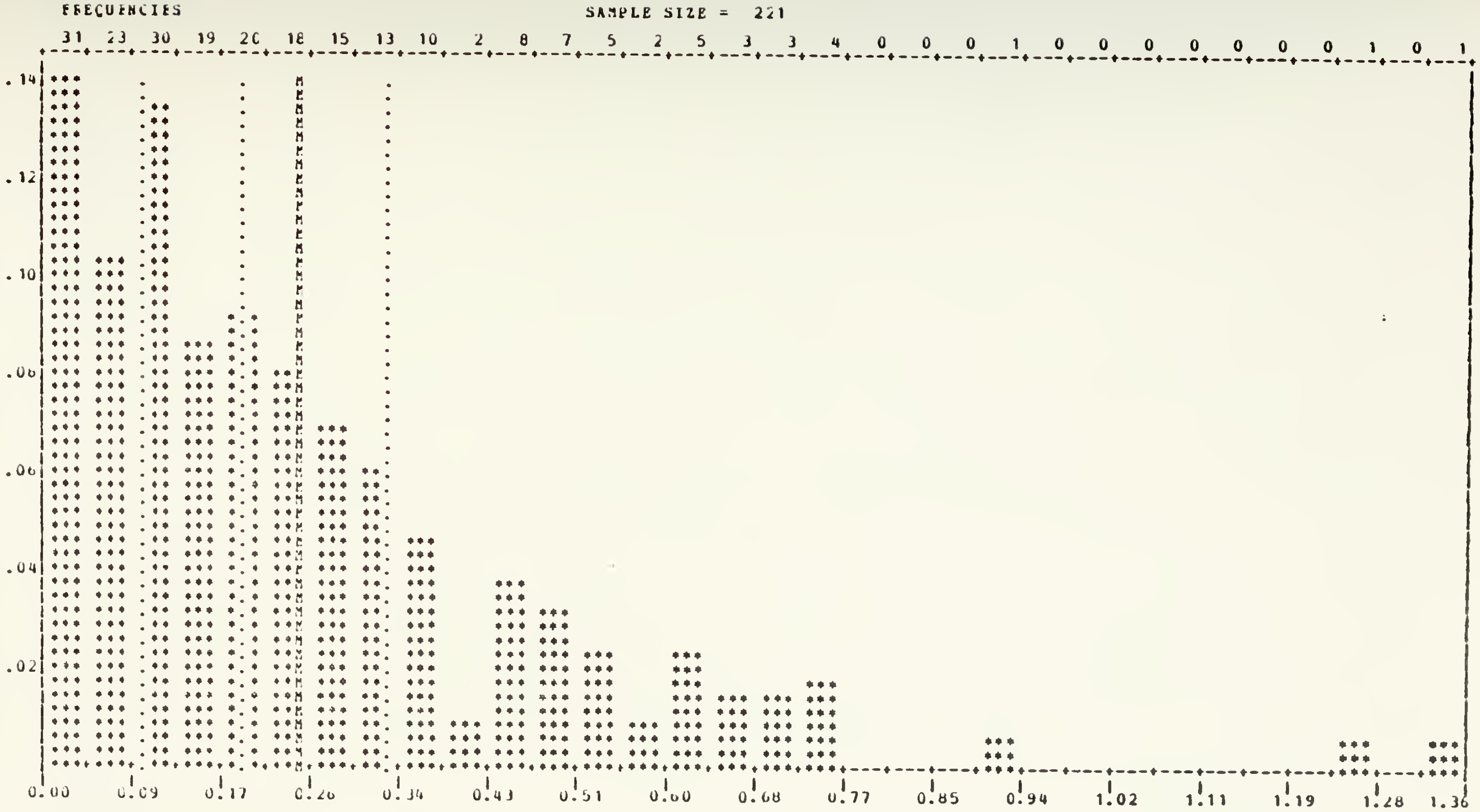
$$A_i = \begin{cases} rA_{i-1} & \text{w.p. } r \\ rA_{i-1} + \mathcal{E}_i RX/RS & \text{w.p. } (1-r) \end{cases}$$

Again we chose the values .25, .50, .95, .99, for t and .25, .50, .90, .95, .98 for r and we run a total of 20 runs to cover all combinations of t and r, naming each run as SC#tt#rr.

Analyzing the results obtained from the simulation we observe the following:

1. The value of the traffic intensity t and the correlation r is the main factor for convergence of W and $\bar{W}$. That is, high traffic intensity and/or high correlation require N to be large in order for W and $\bar{W}$ to reach the steady state. Thus simulating the models with $t \leq .50$ and $r \leq .95$ for $N=10000$ it was possible for W and $\bar{W}$ to reach the steady state. In figures 35a-35d, 36a-36c we can see the justification of that conclusion. The values .1078 and .0957 of W_{10000}, and $\bar{W}_{10000}$ represent the steady state of

the SC#50#25 model. On the other hand, regardless of the value of t the models with r≥.98 did not reach the steady state for N=10000 and regardless of the value of r the models with t≥.95 did not reach the steady state for N=10,000 . Figure 37 gives us the non-convergence of the model SC#99#25. Because of the high traffic intensity and in order to get the steady state of that model we have to continue simulating by increasing the value of N.



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CENTRAL TENDENCY              SPREAD                        HIGHER CENTRAL MOMENTS       DISTRIBUTION

MEAN        2.439452E-01      VARIANCE    4.705509E-02      M3          1.816077E-02     MINIMUM                      1.600000E-03
MEDIAN      1.826000E-01      STD DEV     2.169219E-01      M4          1.696034E-02     .10 QUANTILE                 3.530000E-02
TRIMEAN     1.947624E-01      COEF VAR    8.892238E-01      SKEWNESS    1.779198E 00     .25 QUANTILE  (HINGE)        8.739996E-02
MIDMEAN     1.924022E-01      MEAN DEV    1.555470E-01      KURTOSIS    4.659870E 00     .50 QUANTILE  (MEDIAN)       1.826000E-01
MIDRANGE    6.830492E-01      RANGE       1.362899E 00      BETA1       1.791499E-02     .75 QUANTILE  (HINGE)        3.264499E-01
GEOM MEAN   1.576179E-01      MIDSPREAD   2.390500E-01      BETA2       1.671308E-02     .90 QUANTILE                 5.211999E-01
HARM MEAN   7.074016E-02                                                                 MAXIMUM                      1.364499E 00
```

Figure 35a - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. HISTOGRAM OF THE WAITING TIMES W_{10000} WITHOUT ZEROS FROM THE RUN SC#50#25; m=500 REPLICATIONS, RX=2; RS=4 . No OF ZEROS=279.

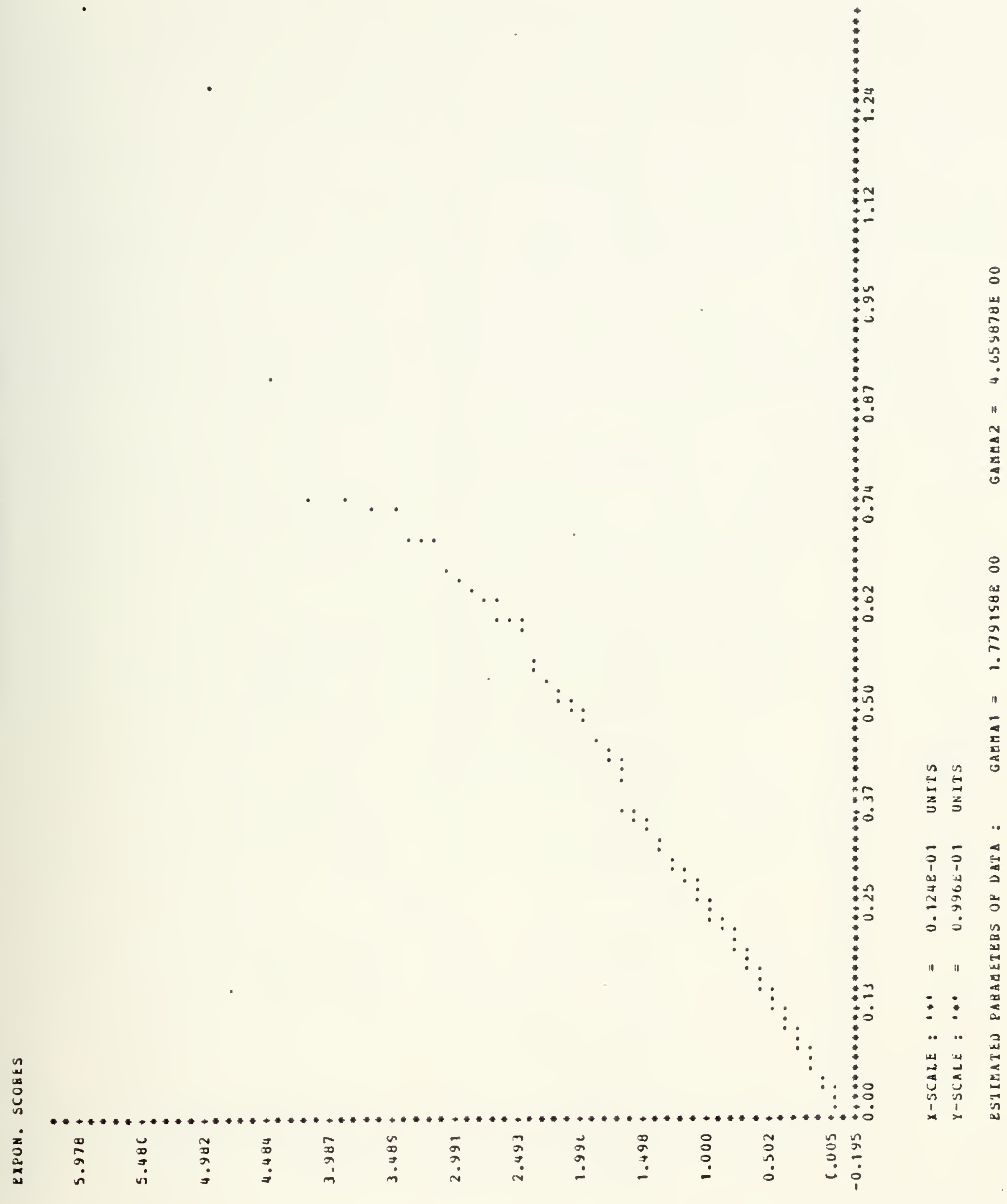


Figure 35b - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. EXPONENTIAL PLOT (EXPLT) OF THE WAITING TIMES W_{10000} WITHOUT ZEROS FROM THE RUN SC#50#25; m=500 REPLICATIONS, RX=2; RS=4 . No OF ZEROS=279.

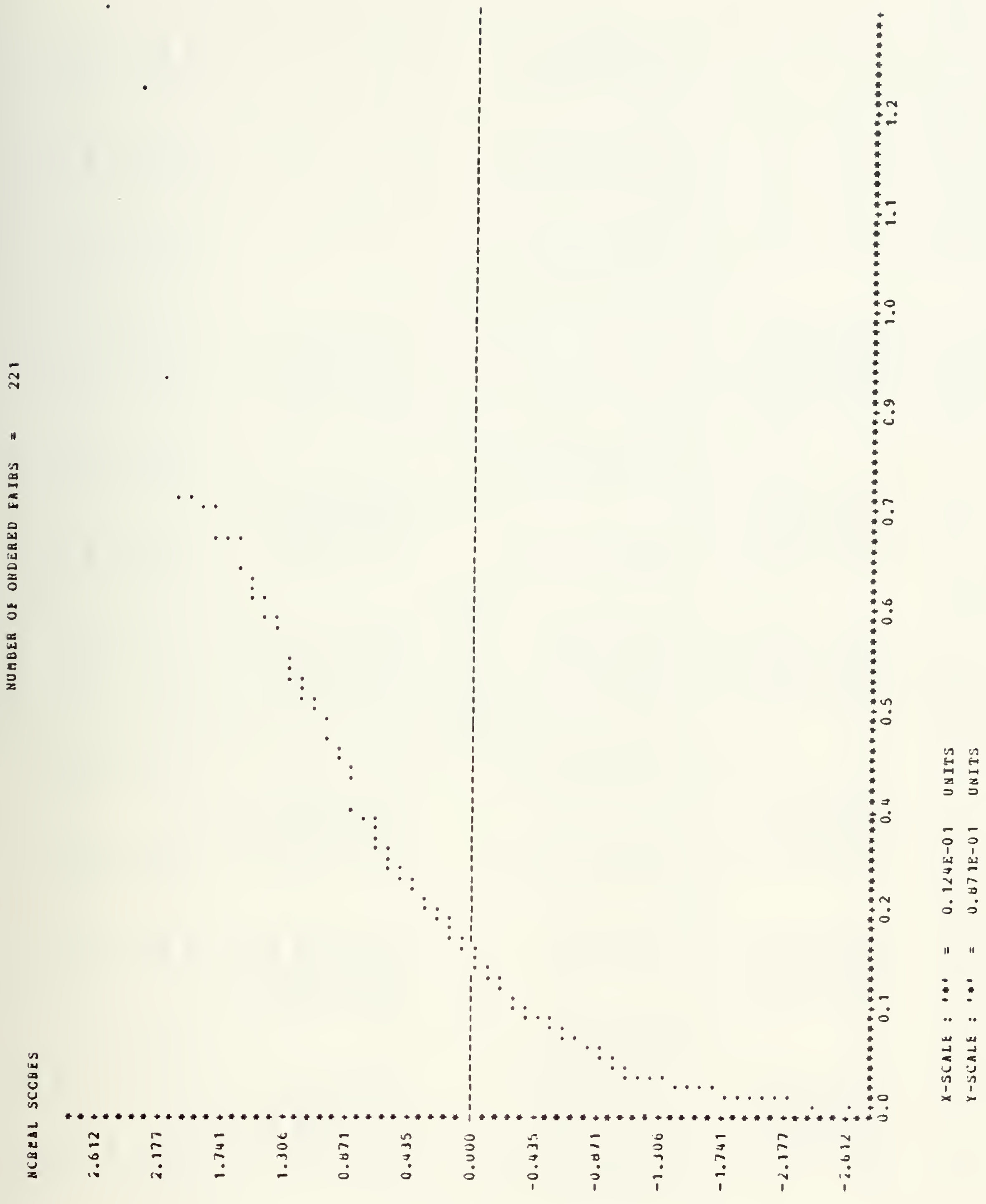


Figure 35c - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. NORMAL PLOT (NORMPL) OF THE WAITING TIMES W_{10000} WITHOUT ZEROS FROM THE RUN SC#50#25; m=500 REPLICATIONS, RX=2; RS=4 . No OF ZEROS=279.

CORRELATED QUEUE

PARMTS	X 2	X 4	S 2	S 4	W 1 + 0S	W 1	W 2 + 0S	W 2	W 3 + 0S	W 3	W 4 + 0S	W 4
MEAN	2499.491	4996.012	1248.523	2498.391	0.09003	0.22174	0.08832	0.21130	0.10627	0.25670	0.10782	0.24395
S(MEAN)	1.5480	2.1660	0.8261	1.1685	0.00729	0.01336	0.00683	0.01195	0.00940	0.01814	0.00842	0.01459
ST. DEV.	34.6141	48.4334	18.4721	26.1284	0.16296	0.19040	0.15270	0.17272	0.21009	0.26098	0.18829	0.21692
SKEWNESS	-0.0260	-0.2050	0.0727	-0.1522	2.16911	1.10451	2.14244	1.23810	2.86387	1.79610	2.53022	1.77920
KURTOS.	0.1793	0.1020	0.1292	-0.1513	4.50274	0.72695	4.59767	1.23196	9.57336	3.54677	8.39917	4.65987
SMPL SIZE	500.0000	500.0000	500.0000	500.0000	500.00000	203.00000	500.00000	209.00000	500.00000	207.00000	500.00000	221.00000

PARMTS	W 1 BAR	W 2 BAR	W 3 BAR	W 4 BAR	D 2	D 4
MEAN	0.0953327	0.0953571	0.0957121	0.0957863	-0.0001937	0.0002375
S(MEAN)	0.0003439	0.0002440	0.0002007	0.0001769	0.0002694	0.0001920
ST. DEV.	0.0076895	0.0054555	0.0044877	0.0039563	0.0060232	0.0042922
SKEWNESS	0.6068329	0.3033307	0.1048896	0.1783530	0.1998998	0.0720455
KURTOS.	0.7583609	0.2003317	-0.1096115	-0.1895771	0.4037561	0.0656462
SMPL SIZE	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000

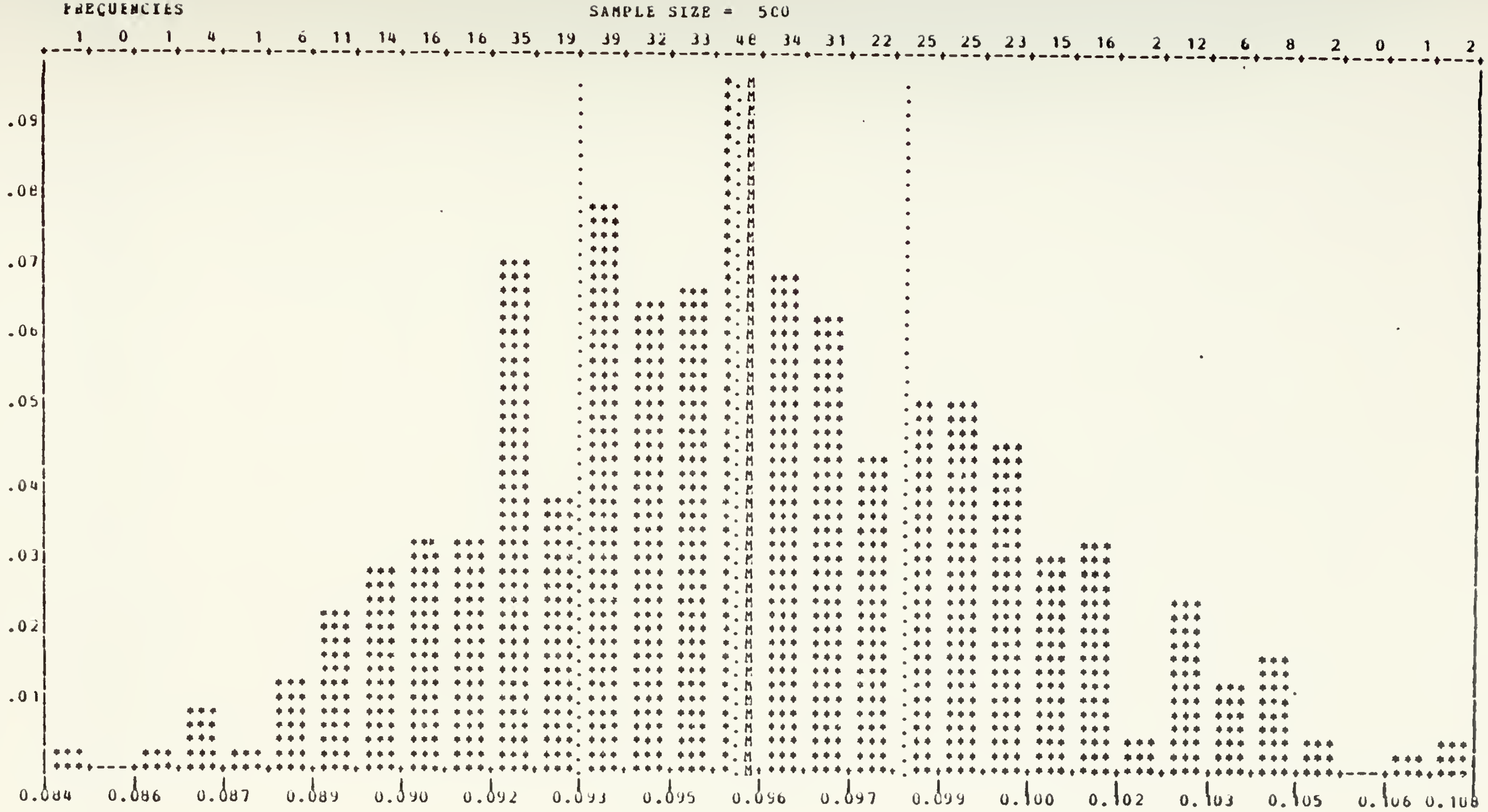
UNCORRELATED QUEUE

PARMTS	X 2	X 4	S 2	S 4	W 1 + 0S	W 1	W 2 + 0S	W 2	W 3 + 0S	W 3	W 4 + 0S	W 4
MEAN	2499.491	4996.012	1248.937	2498.015	0.24793	0.46255	0.22471	0.44585	0.27123	0.57709	0.27664	0.50852
S(MEAN)	1.5480	2.1660	0.7893	1.1556	0.01826	0.02812	0.01633	0.02567	0.02366	0.04224	0.02037	0.03113
ST. DEV.	34.6141	48.4334	17.6482	25.8401	0.40829	0.46032	0.36516	0.40755	0.52894	0.64759	0.45542	0.51337
SKEWNESS	-0.0260	-0.2050	0.2370	0.0945	2.27934	1.60777	2.11334	1.39835	3.09001	2.15518	2.87300	2.38006
KURTOS.	0.1783	0.1020	0.0275	0.0293	5.69149	2.64647	4.49550	1.63665	12.53415	6.30046	13.30202	10.00915
SMPL SIZE	500.0000	500.0000	500.0000	500.0000	500.00000	268.00000	500.00000	252.00000	500.00000	235.00000	500.00000	272.00000

PARMTS	W 1 BAR	W 2 BAR	W 3 BAR	W 4 BAR	D 2	D 4
MEAN	0.2484031	0.2486218	0.2489119	0.2488778	-0.0001108	0.0002000
S(MEAN)	0.0011608	0.0008263	0.0007043	0.0006207	0.0003462	0.0002467
ST. DEV.	0.0259569	0.0184774	0.0157478	0.0138790	0.0077402	0.0055169
SKEWNESS	0.4764770	0.3973002	0.1904184	0.3035822	0.1977163	0.2697002
KURTOS.	0.0193195	0.0144081	0.1101370	0.0134916	-0.0576601	-0.0075665
SMPL SIZE	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000	500.0000000

Figure 35d - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. TABULATION OF SAMPLE STATISTICS FOR THE DISTRIBUTIONS OF CUMULATED INTERARRIVAL TIMES AT N=5000 AND N=10000 (X 2 AND X 4), CUMULATED SERVICE TIMES (S 2 AND S 4), WAITING TIMES, WITH OR WITHOUT ZEROS (WI + 0, WI, I FOR CASE N=2500, 5000, 7500, 10000), CUMULATED WAITING TIMES (W1 BAR etc.) AND THE AVERAGED DIFFERENCES BETWEEN S2 AND X2 (D2); OBTAINED FROM m=500 REPLICATIONS OF THE RUN SC#50#25; RX=2; RS=4 .

2. High trafic intensity and correlation affects the convergence of the distribution of W and $\bar{W}$ also. Furthermore the steady state of their distributions, requires larger N than is required for the steady state of their expected values. In examining that conclusion, look for example at the results of the simulated model SC#50#98, where although W and $\bar{W}$ have already reached the steady state for $N=10000$, their distribution has not converged yet. The above results have been obtained from Figures 38a-38c, 39a-39d where W_{10000} and $\bar{W}_{10000}$ are presented. We can see from these figures (38a-38c) that although the value of W_{10000} has reached the steady state (see Figure 39d), its distribution cannot be characterized since the plot under EXPIT (figure 38b) is not a straight line and the parameters $\gamma_L=3.34$ and $\gamma_L=11.0$ are far away from the desired values 2 and 6 . The same results can be obtained from the figures 39a-39c which deal with $\bar{W}_{10000}$.



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CENTRAL TENDENCY            SPREAD                        HIGHER CENTRAL MOMENTS       DISTRIBUTION

MEAN        9.578633E-02    VARIANCE   1.565226E-05       M3          1.104448E-08     MINIMUM                          8.429998E-02
MEDIAN      9.559995E-02    STD DEV    3.956292E-03       M4          6.885343E-10     .10 QUANTILE                     9.069997E-02
TRIMEAN     9.569991E-02    COEF VAR   4.130330E-02       SKEWNESS    1.783530E-01     .25 QUANTILE  (HINGE)            9.309995E-02
MIDMEAN     9.563315E-02    MEAN DEV   3.151199E-03       KURTOSIS   -1.895771E-01     .50 QUANTILE  (MEDIAN)           9.559995E-02
MIDRANGE    9.599996E-02    RANGE      2.340001E-02       BETA1       1.097830E-08     .75 QUANTILE  (HINGE)            9.849995E-02
GEOM MEAN   9.570515E-02    MIDSPREAD  5.400002E-03       BETA2       6.859571E-10     .90 QUANTILE                     1.010000E-01
HARM MEAN   9.562397E-02                                                               MAXIMUM                          1.077000E-01
```

Figure 36a - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. HISTOGRAM OF THE CUMULATED AND AVERAGED WAITING TIMES $\bar{W}_{10000}$ FROM THE RUN SC#50#25; m=500 REPLICATIONS, RX=2; RS=4 .

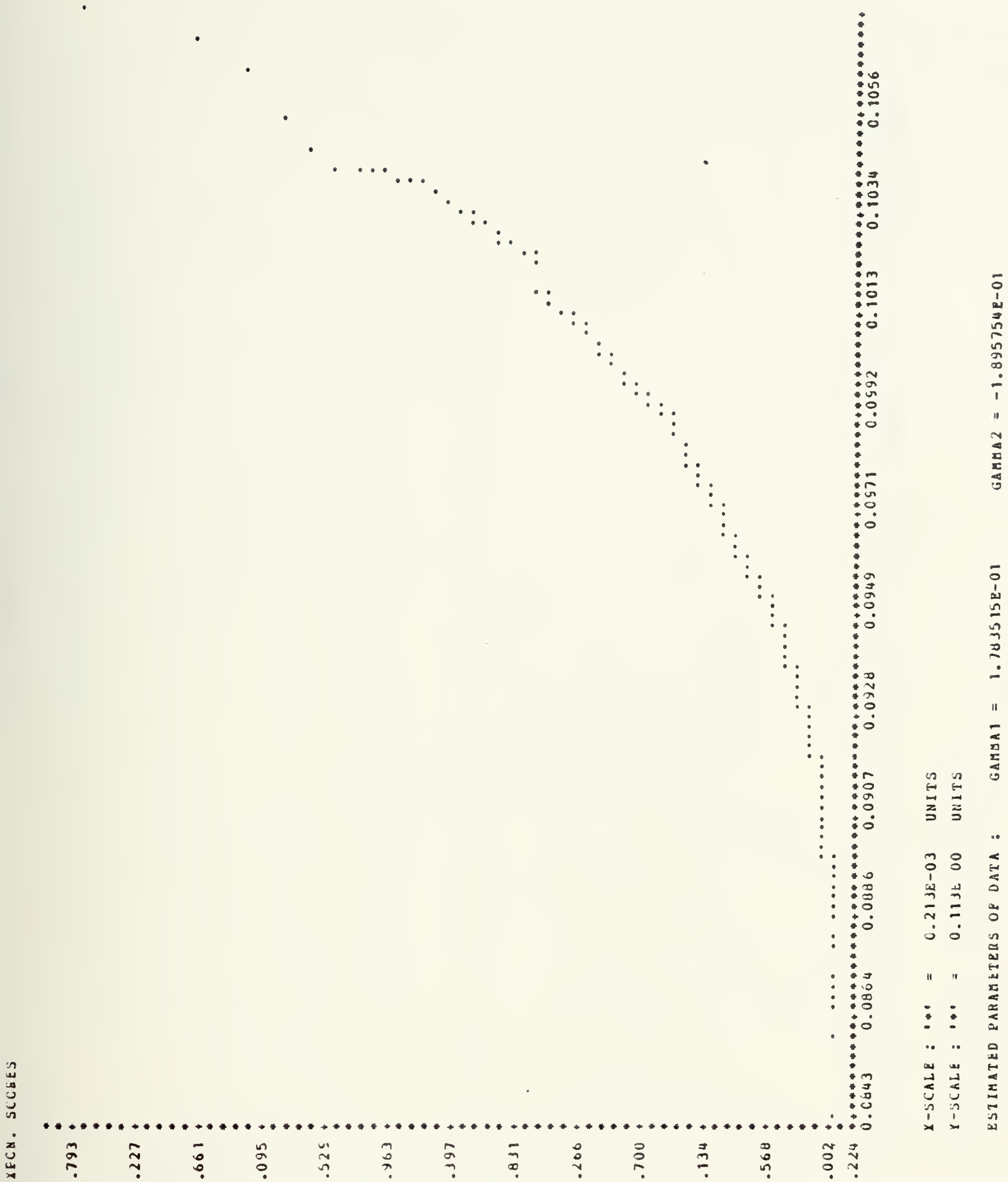


Figure 36b - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. EXPONENTIAL PLOT (EXPLT) OF THE CUMULATED AND AVERAGED WAITING TIMES $\bar{W}_{10000}$ FROM THE RUN SC#50#25; m=500 REPLICATIONS, RX=2; RS=4 .

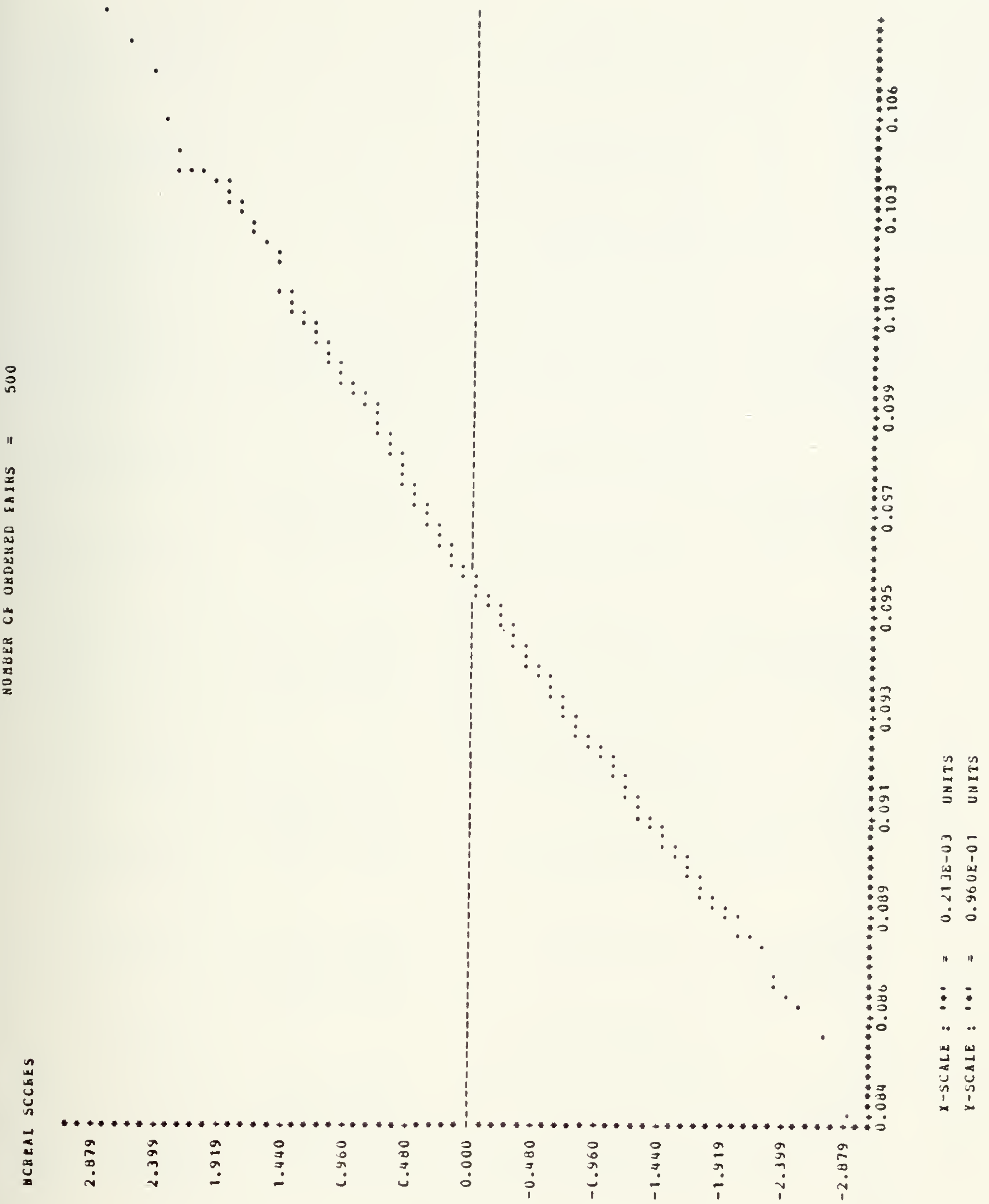
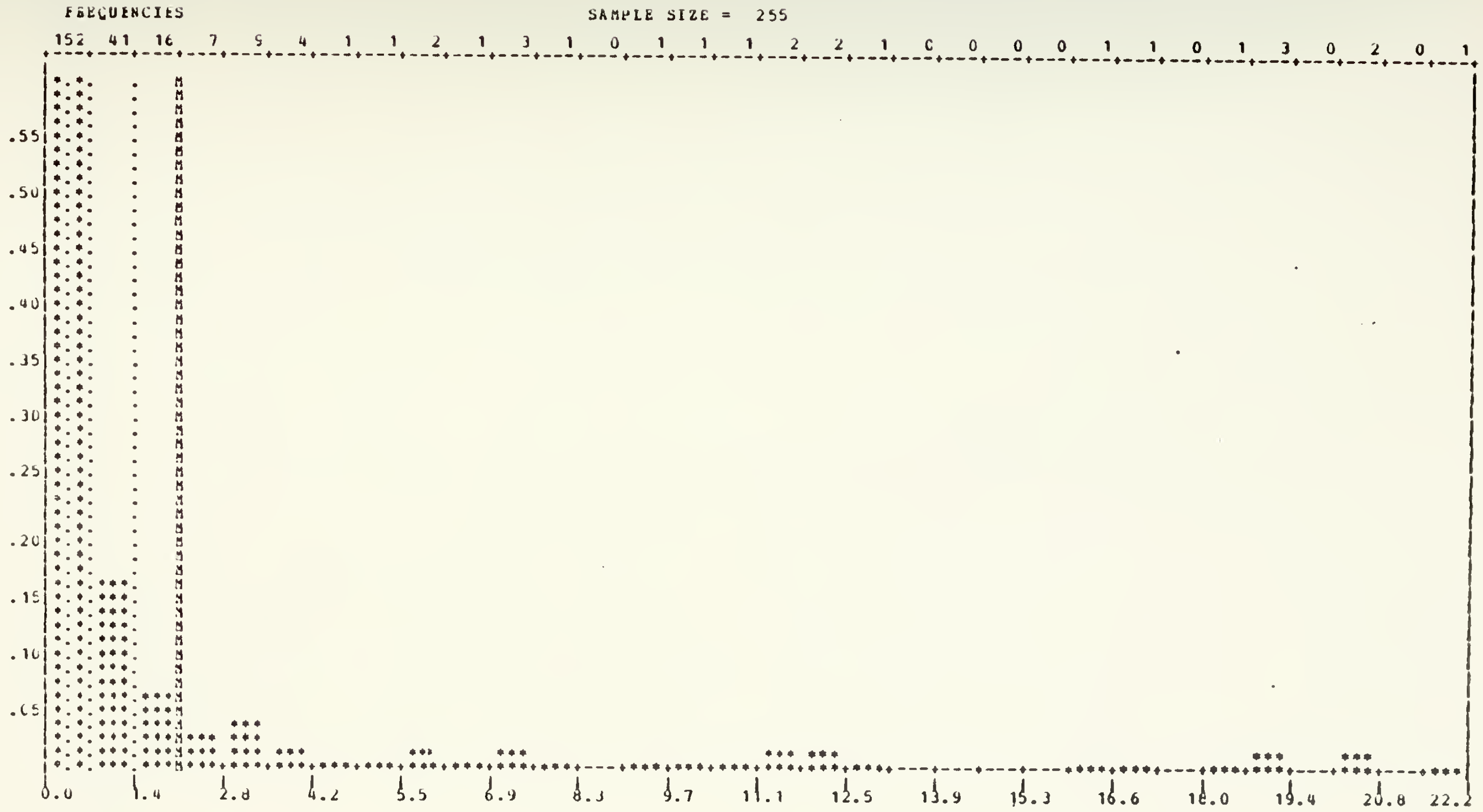


Figure 36c - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. NORMAL PLOT (NORMPL) OF THE CUMULATED AND AVERAGED WAITING TIMES $\bar{W}_{10000}$ FROM THE RUN SC#50#25; m=500 REPLICATIONS, RX=2; RS=4 .

```
                                                     CORRELATED QUEUE

 PARMTR      X 2       X 4       S 2       S 4    W 1 + CS     W 1    W 2 + CS     W 2    W 3 + 0S     W 3    W 4 + 0S     W 4

 MEAN      1683.007  3366.192  1664.831  3331.525  10.19420  10.29717  12.90386  13.06733  14.64498  14.82266  15.51440  15.60805
 S(MEAN)      1.0424    1.4631    1.1015    1.5580   0.38230   0.38337   0.50402   0.50640   0.57637   0.57925   0.60965   0.61092
 ST. DEV.    23.3082   32.7162   24.6300   34.8388   8.54841   8.52951  11.27030  11.24385  12.89921  12.87528  13.63212  13.61962
 SKEWNESS    -0.0261   -0.2053    0.0728   -0.1522   1.30525   1.30529   1.55715   1.56307   1.24912   1.24782   1.43827   1.43925
 KURTOS.      0.1783    0.1018    0.1293   -0.1511   2.40135   2.41711   3.29545   3.32156   1.57342   1.57325   2.65161   2.65871
 SMPL SIZE  500.0000  500.0000  500.0000  500.0000 500.00000 495.00000 500.00000 493.00000 500.00000 454.00000 500.00000 457.00000

 PARMTR         W 1  BAR       W 2  BAR       W 3  BAR       W 4  BAR         D 2            D 4

 MEAN          7.2342557      9.3939286     10.9133062     11.8662443     -0.0002662     -0.0000597
 S(MEAN)       0.1872380      0.2473476      0.2757985      0.2974685      0.0002158      0.0001559
 ST. DEV.      4.1867714      5.5508628      6.1670446      6.6520491      0.0048252      0.0034870
 SKEWNESS      1.7057558      2.0456374      1.9305573      1.8245697      0.1376542     -0.1320593
 KURTOS.       4.5712099      6.4233065      5.6494434      5.2069817      0.1230812      0.0209312
 SMPL SIZE   500.0000000    500.0000000    500.0000000    500.0000000    500.0000000    500.0000000


                                                     UNCORRELATED QUEUE

 PARMTR      X 2       X 4       S 2       S 4    W 1 + CS     W 1    W 2 + CS     W 2    W 3 + 0S     W 3    W 4 + 0S     W 4

 MEAN      1683.007  3366.192  1663.384  3321.025  14.82400  15.03446  16.92291  17.12697  21.96506  22.27755  23.66727  23.90634
 S(MEAN)      1.0424    1.4631    1.0524    1.5408   0.56661   0.56925   0.75766   0.76221   0.91403   0.91937   1.02615   1.03093
 ST. DEV.    23.3082   32.7162   23.5316   34.4538  12.67426  12.63930  16.94191  16.90674  20.43829  20.41335  22.94553  22.93684
 SKEWNESS    -0.0261   -0.2053    0.2369    0.0945   1.14059   1.13940   1.44110   1.44187   1.34763   1.34367   1.90339   1.90466
 KURTOS.      0.1783    0.1018    0.0275    0.0294   0.79620   0.79072   2.37382   2.37581   1.81030   1.80023   5.34096   5.34691
 SMPL SIZE  500.0000  500.0000  500.0000  500.0000 500.00000 493.00000 500.00000 492.00000 500.00000 493.00000 500.00000 495.00000

 PARMTR         W 1  BAR       W 2  BAR       W 3  BAR       W 4  BAR         D 2            D 4

 MEAN         10.3516266     13.8062275     16.1548462     17.7791595     -0.0001574     -0.0001499
 S(MEAN)       0.2757736      0.3830756      0.4493776      0.5010985      0.0002946      0.0002138
 ST. DEV.      6.2559299      8.5658340     10.0416851     11.2049074      0.0065921      0.0047798
 SKEWNESS      1.4525485      1.4735689      1.6260328      1.7394295      0.2896590      0.2765061
 KURTOS.       1.9141411      2.0772343      2.9112806      3.5989559     -0.1155488      0.1787348
 SMPL SIZE   500.0000000    500.0000000    500.0000000    500.0000000    500.0000000    500.0000000
```

Figure 37 - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. TABULATION OF SAMPLE STATISTICS FOR THE DISTRIBUTIONS OF CUMULATED INTERARRIVAL TIMES AT N=5000 AND N=10000 (X 2 AND X 4), CUMULATED SERVICE TIMES (S 2 AND S 4), WAITING TIMES, WITH OR WITHOUT ZEROS (WI + 0, WI, I FOR CASE N=2500, 5000, 7500, 10000), CUMULATED WAITING TIMES (WI BAR etc.) AND THE AVERAGED DIFFERENCES BETWEEN S2 AND X2 (D2); OBTAINED FROM m=500 REPLICATIONS OF THE RUN SC#99#25; RX=2.97; RS=3 .



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CENTRAL TENDENCY              SPREAD                        HIGHER CENTRAL MOMENTS        DISTRIBUTION

MEAN          1.905525E 00    VARIANCE     1.609851E 01    M3           2.160332E 02    MINIMUM                       2.000000E-04
MEDIAN        5.085000E-01    STD DEV      4.012295E 00    M4           3.627249E 03    .10 QUANTILE                  7.949996E-02
TRIMEAN       6.140497E-01    COEF VAR     2.105611E 00    SKEWNESS     3.344586E 00    .25 QUANTILE  (HINGE)         1.698999E-01
MIDMEAN       5.413976E-01    MEAN DEV     1.709286E 00    KURTOSIS     1.099608E 01    .50 QUANTILE  (MEDIAN)        5.065000E-01
MIDRANGE      1.109104E 01    RANGE        2.218167E 01    BETA1        2.134983E 02    .75 QUANTILE  (HINGE)         1.269300E 00
GEOM MEAN     5.030937E-01    MIDSPREAD    1.099400E 00    BETA2        3.576415E 03    .90 QUANTILE                  3.945200E 00
HARM MEAN     3.840891E-02                                                              MAXIMUM                       2.218188E 01
```

Figure 38a - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. HISTOGRAM OF THE WAITING TIMES W_{10000} WITHOUT ZEROS FROM THE RUN SC#50#98; m=500 REPLICATIONS, RX=2; RS=4 . No OF ZEROS=245.

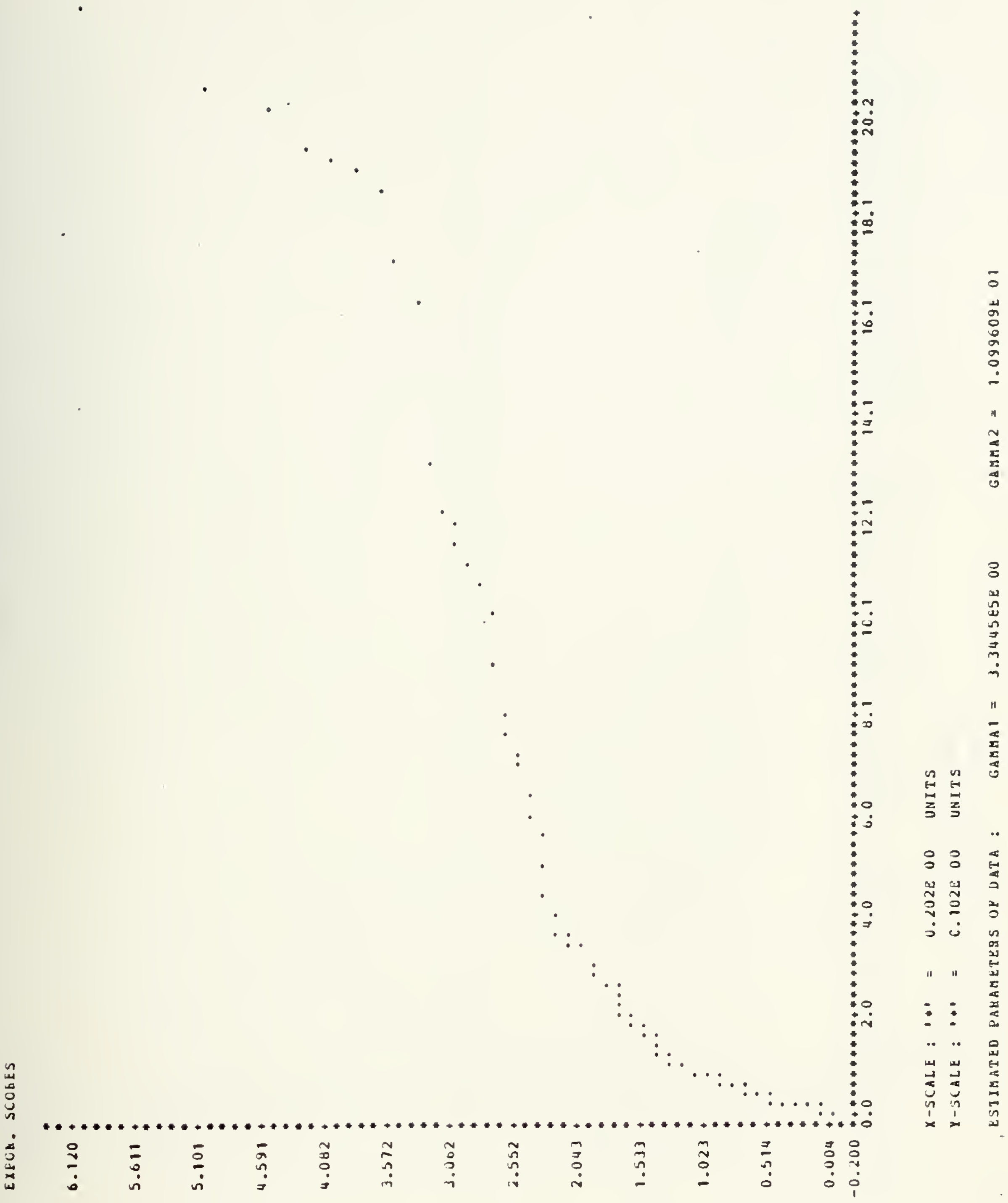


Figure 38b - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. EXPONENTIAL PLOT (EXPLT) OF THE WAITING TIMES W_{10000} WITHOUT ZEROS FROM THE RUN SC#50#98; m=500 REPLICATIONS, RX=2; RS=4 . No OF ZEROS=245.

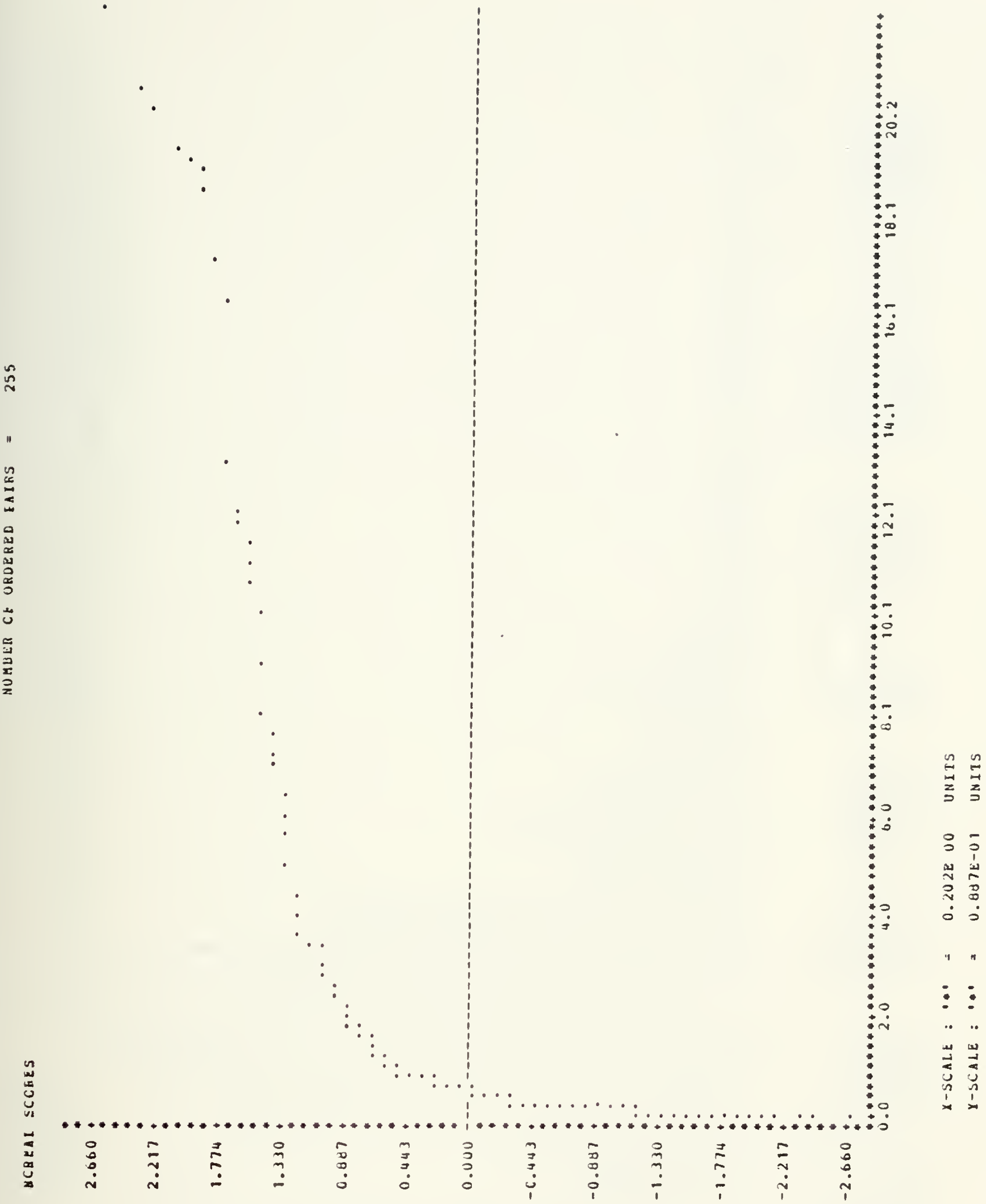
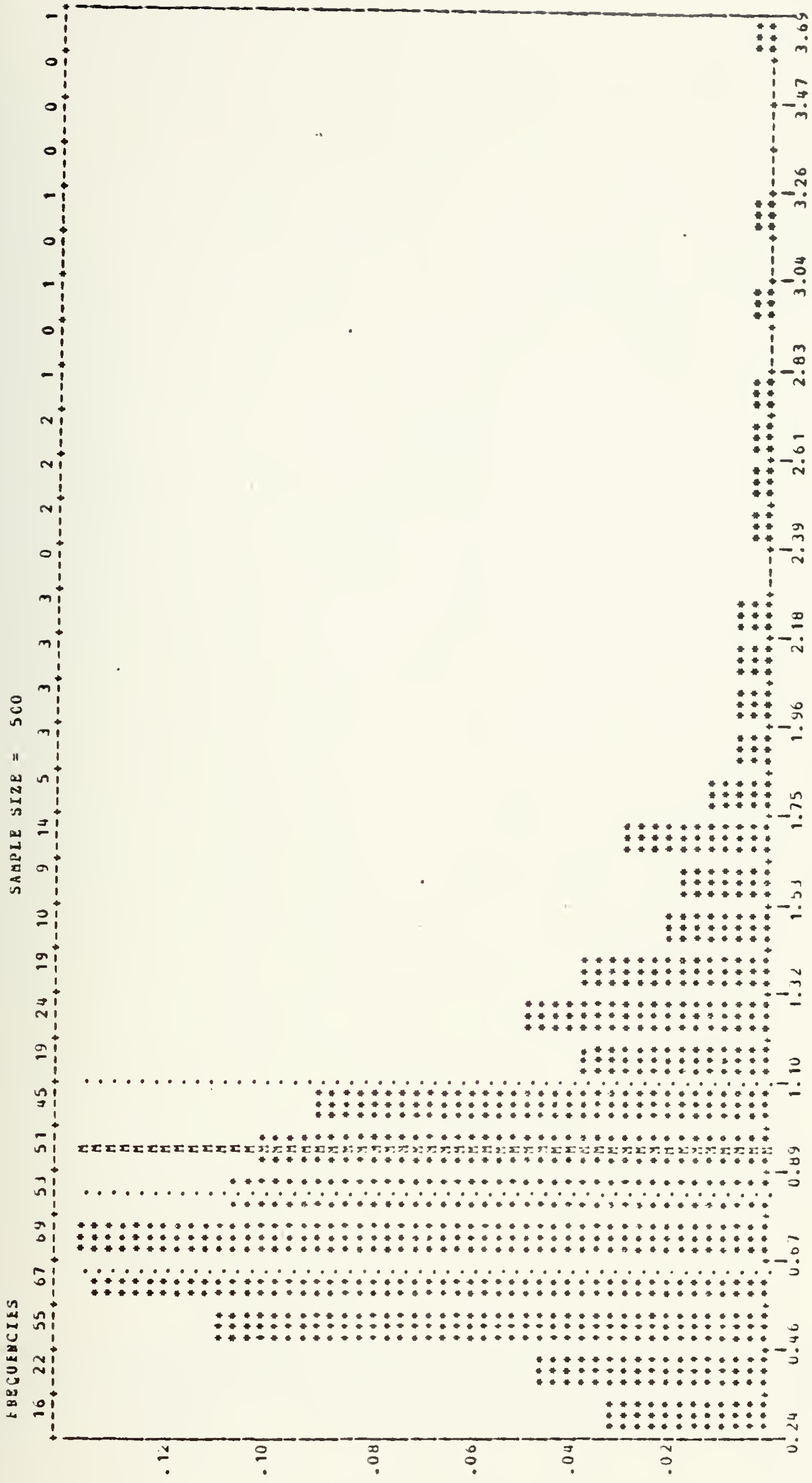


Figure 38c - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. NORMAL PLOT (NORMPL) OF THE WAITING TIMES W_{10000} WITHOUT ZEROS FROM THE RUN SC#50#98; m=500 REPLICATIONS, RX=2; RS=4 . No OF ZEROS=245.



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CENTRAL TENDENCY             SPREAD                      HIGHER CENTRAL MOMENTS       DISTRIBUTION
MEAN       9.299122E-01      VARIANCE   2.214087E-01     M3         1.877570E-01      MINIMUM                  2.390000E-01
MEDIAN     8.121996E-01      STD DEV    4.705408E-01     M4         3.878805E-01      .10 QUANTILE             4.782000E-01
TRIMEAN    8.328245E-01      COEF VAR   5.060056E-01     SKEWNESS   1.802206E 00      .25 QUANTILE (HINGE)     6.138496E-01
MIDMEAN    8.326610E-01      MEAN DEV   3.292990E-01     KURTOSIS   4.912409E 00      .50 QUANTILE (MEDIAN)    8.121996E-01
MIDRANGE   1.963900E 00      RANGE      3.447999E 00     BETA1      1.866320E-01      .75 QUANTILE (HINGE)     1.093049E 00
GEOM MEAN  8.352572E-01      MIDSPREAD  4.791994E-01     BETA2      3.853640E-01      .90 QUANTILE             1.530199E 00
HARM MEAN  7.552598E-01                                                               MAXIMUM                  3.687900E 00
```

Figure 39a - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. HISTOGRAM OF THE CUMULATED AND AVERAGED WAITING TIMES $\overline{W}_{10000}$ FROM THE SC#50#98; m=500 REPLICATIONS, RX=2; RS=4 .

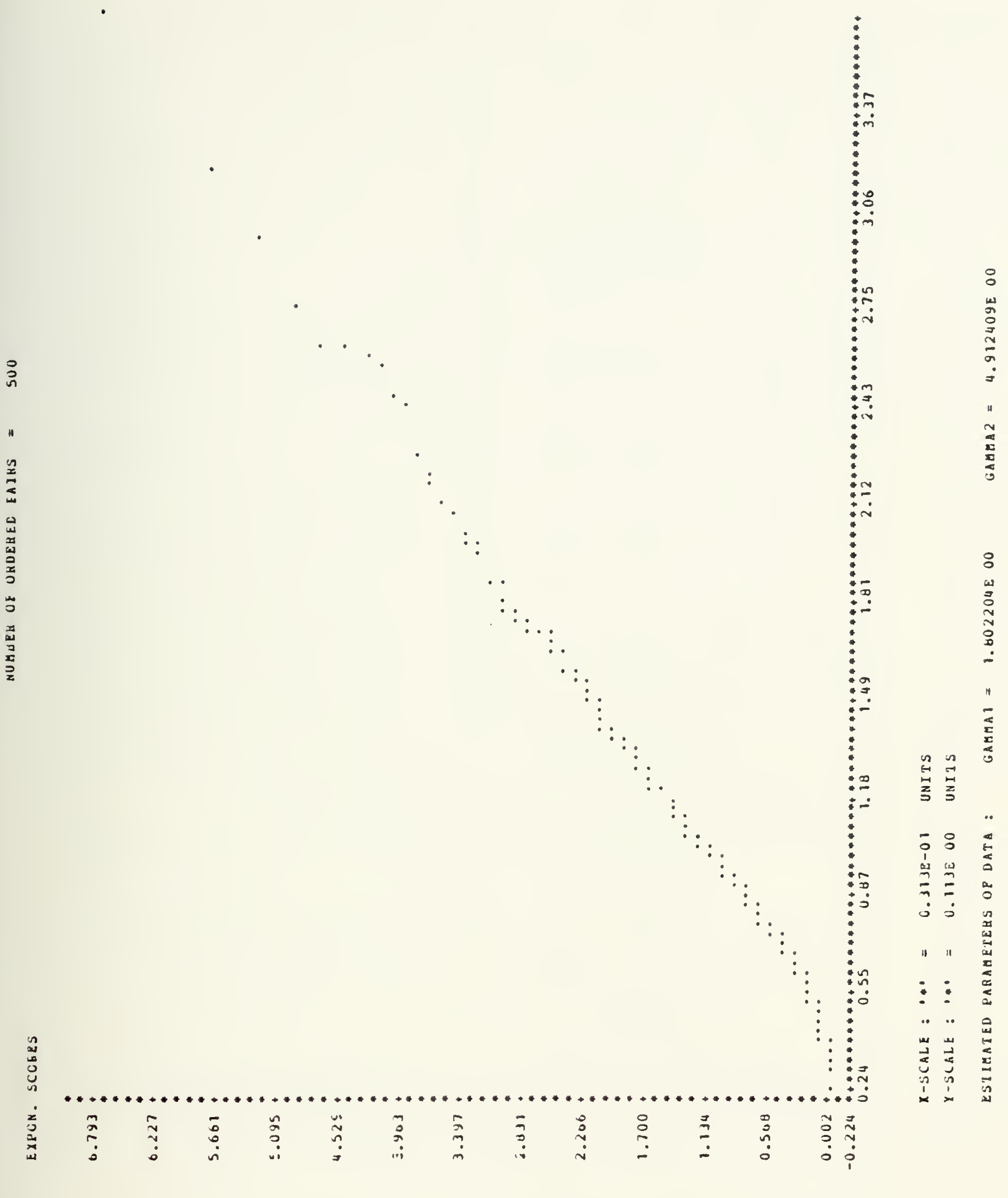


Figure 39b - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. EXPONENTIAL PLOT (EXPIT) OF THE CUMULATED AND AVERAGED WAITING TIMES $\bar{W}_{10000}$ FROM THE SC#50#98; n=500 REPLICATIONS, RX=2; RS=4 .

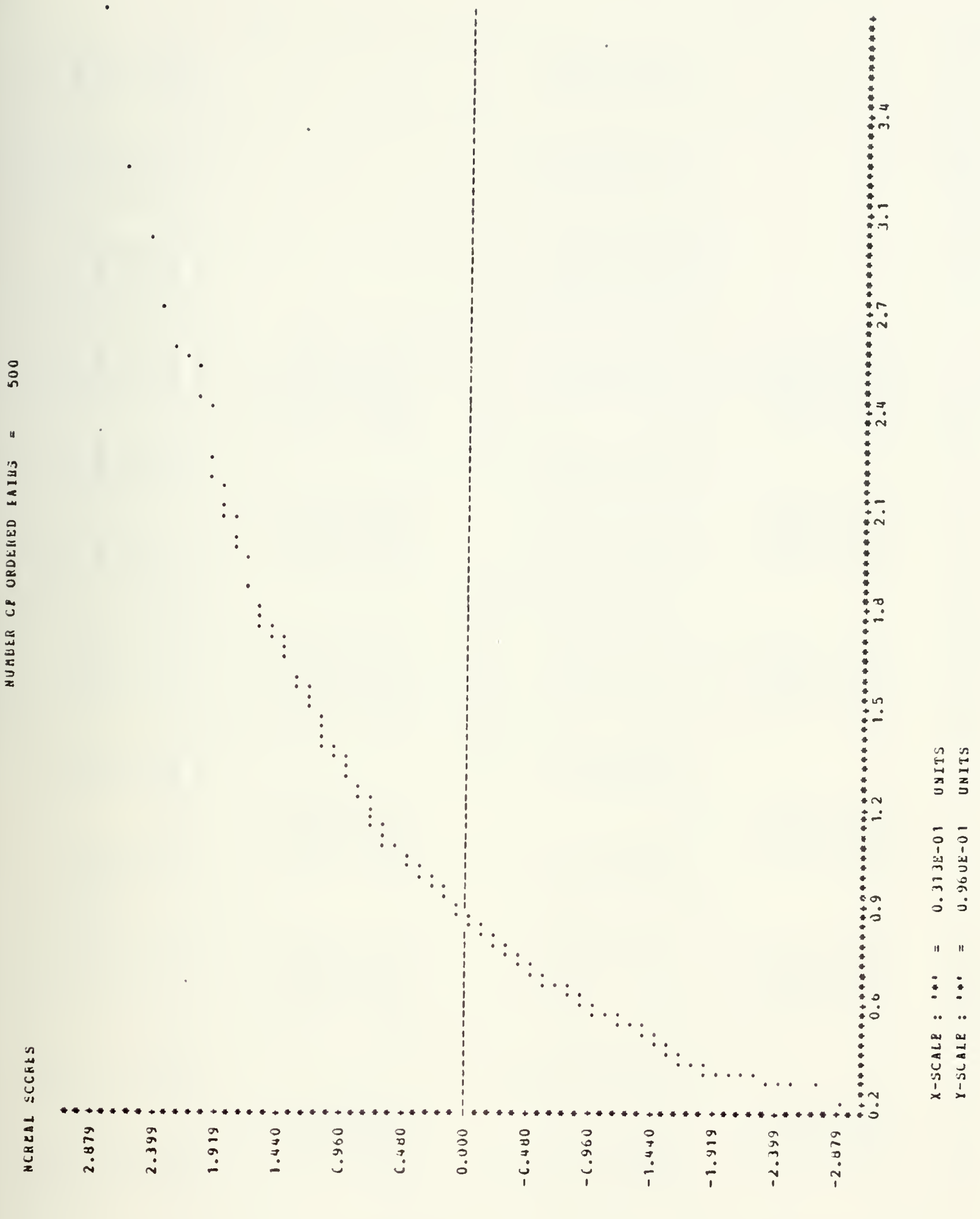


Figure 39c - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. NORMAL PLOT (NORMPL) OF THE CUMULATED AND AVERAGED WAITING TIMES $\bar{W}_{10000}$ FROM THE SC#50#98; m=500 REPLICATIONS, RX=2; RS=4 .

```
                                                  CORRELATED QUEUE

PARMTR       X 2       X 4       S 2       S 4     W 1 + 0S    W 1     W 2 + 0S    W 2     W 3 + 0S    W 3     W 4 + 0S    W 4

MEAN      2459.491  4996.012  1253.389  2501.876   0.82910   1.77159   0.78178   1.53894   1.01661   2.15383   0.97182   1.90553
S(MEAN)      1.5480    2.1660    3.8788    5.7193   0.15756   0.32625   0.13082   0.24868   0.15299   0.30802   0.13493   0.25126
ST. DEV.    34.6141   48.4334   86.7337  127.8872   3.52316   4.99063   2.92525   3.96327   3.42097   4.73185   3.01722   4.01229
SKEWNESS    -0.0260   -0.2050    0.2924    0.2197   9.29953   6.46620  10.06820   7.43863   6.11774   4.20099   4.82571   3.34459
KURTOS.      0.1783    0.1020    0.1695    0.0154 108.90057  51.79210 141.44316  76.10095  46.89284  21.70930  24.65054  10.99608
SMPL SIZE  500.0000  500.0000  500.0000  500.0000 500.00000 234.00000 500.00000 254.00000 500.00000 236.00000 500.00000 255.00000

PARMTR       W 1  BAR      W 2  BAR      W 3  BAR      W 4  BAR        D 2           D 4

MEAN        0.9364662     0.9259205     0.9456846     0.9299122     0.0007794     0.0005859
S(MEAN)     0.0460532     0.0299828     0.0245545     0.0210432     0.0008164     0.0005802
ST. DEV.    1.0297813     0.6704358     0.5490551     0.4705408     0.0182555     0.0129748
SKEWNESS    4.7968645     3.0633430     2.0192347     1.8022060     0.1497339     0.1467554
KURTOS.    32.0446930    15.3456116     6.8067226     4.9124088    -0.0639658    -0.0135975
SMPL SIZE 500.0000000   500.0000000   500.0000000   500.0000000   500.0000000   500.0000000


                                                  UNCORRELATED QUEUE

PARMTR       X 2       X 4       S 2       S 4     W 1 + 0S    W 1     W 2 + 0S    W 2     W 3 + 0S    W 3     W 4 + 0S    W 4

MEAN      2459.491  4996.012  1248.937  2498.015   0.24793   0.46255   0.22471   0.44585   0.27123   0.57709   0.27664   0.50852
S(MEAN)      1.5480    2.1660    0.7893    1.1556   0.01826   0.02812   0.01633   0.02567   0.02366   0.04224   0.02037   0.03113
ST. DEV.    34.6141   48.4334   17.6482   25.8401   0.40829   0.46032   0.36516   0.40755   0.52894   0.64759   0.45542   0.51337
SKEWNESS    -0.0260   -0.2050    0.2370    0.0945   2.27934   1.60777   2.11334   1.39835   3.09001   2.15518   2.87300   2.38006
KURTOS.      0.1783    0.1020    0.0275    0.0293   5.69149   2.64647   4.49550   1.63665  12.53415   6.30046  13.30202  10.00915
SMPL SIZE  500.0000  500.0000  500.0000  500.0000 500.00000 268.00000 500.00000 252.00000 500.00000 235.00000 500.00000 272.00000

PARMTR       W 1  BAR      W 2  BAR      W 3  BAR      W 4  BAR        D 2           D 4

MEAN        0.2484031     0.2486218     0.2489119     0.2488778    -0.0001108     0.0002000
S(MEAN)     0.0011608     0.0008263     0.0007043     0.0006207     0.0003462     0.0002467
ST. DEV.    0.0259569     0.0184774     0.0157478     0.0138790     0.0077402     0.0055169
SKEWNESS    0.4764770     0.3973002     0.1904184     0.3035822     0.1977163     0.2697002
KURTOS.     0.0193195     0.0144081     0.1101370     0.0134916    -0.0576601    -0.0075665
SMPL SIZE 500.0000000   500.0000000   500.0000000   500.0000000   500.0000000   500.0000000
```

Figure 39d - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. TABULATION OF SAMPLE STATISTICS FOR THE DISTRIBUTIONS OF CUMULATED INTERARRIVAL TIMES AT N=5000 AND N=10000 (X 2 AND X 4), CUMULATED SERVICE TIMES (S 2 AND S 4), WAITING TIMES, WITH OR WITHOUT ZEROS (WI + 0, WI, I FOR CASE N=2500, 5000, 7500, 10000), CUMULATED WAITING TIMES (W1 BAR etc.) AND THE AVERAGED DIFFERENCES BETWEEN S2 AND X2 (D2); OBTAINED FROM m=500 REPLICATIONS OF THE RUN SC#50#58; EX=1.5; RS=6 .

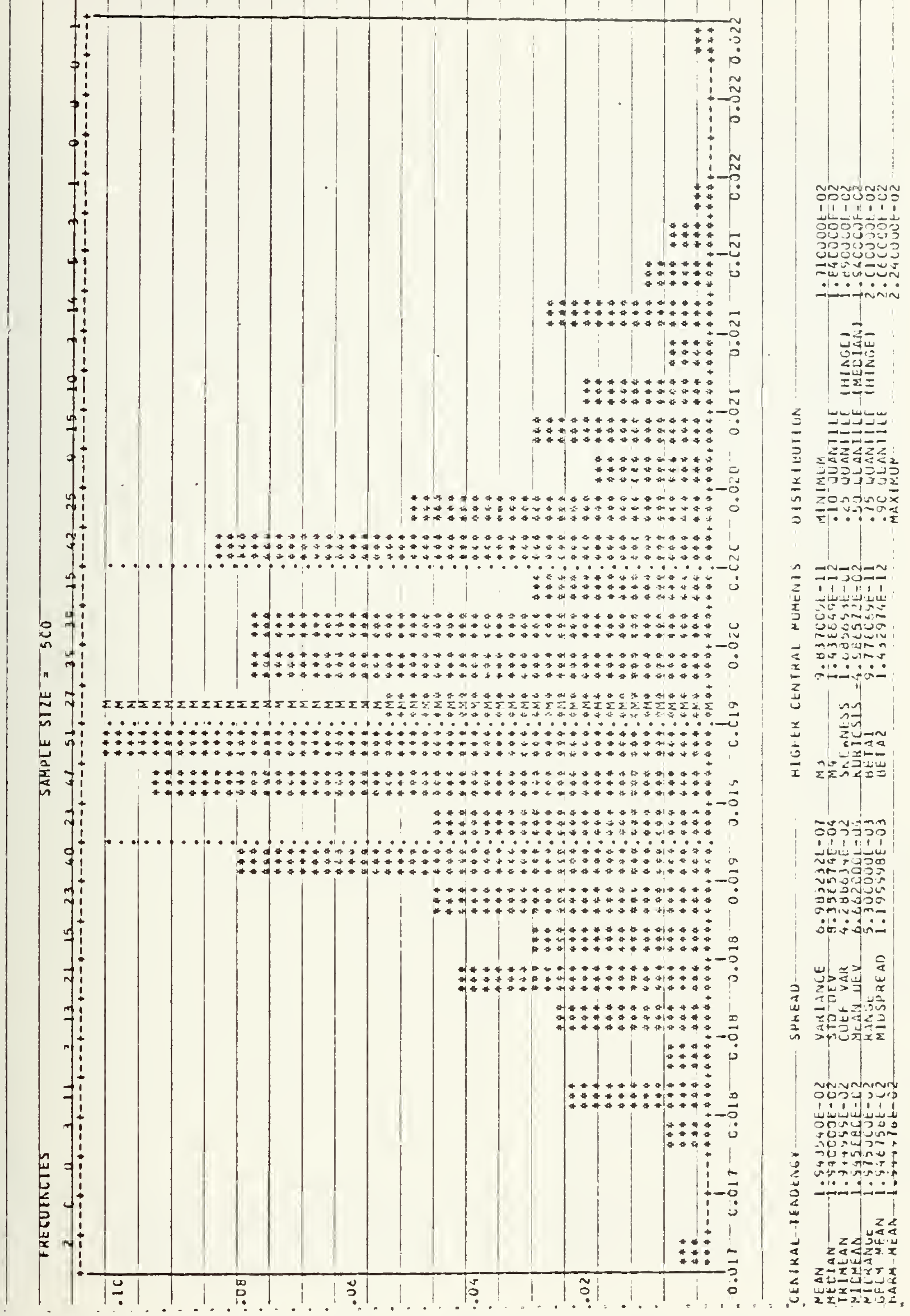
3. To test the distribution of W and $\bar{W}$ we also used the plotting subroutines and the values of their skewness and kurtosis parameters as we did with the queue with dependent service.

The results obtained from the analysis are as follows:

(i). Given that $W>0$, we may say that W has an exponential form, since the plots under EXPLT Subroutine result in a straight line and the values of γ_1 and γ_2 are around their actual values 2 and 6 respectively. For example the SC#50#25 model (which is represented by the figures 35a-35d), where the steady state has been reached, gave us a straight line and the values 1.8 and 4.7 for γ_1 and γ_2 respectively. We have to state here that the exponential distribution comes only if W has reached the steady state, and we cannot have the same result if the W is still in a transient state.

(ii) For the distribution of $\bar{W}$ we can say that the normality assumption is an appropriate one, since the graphs obtained under NORMPL result in a straight line and the value of skewness and kurtosis is around '0'. The results, for example, obtained from the SC#25#25 model gave us a straight line under NORMPL and the values .17 and -.05 for the skewness and kurtosis respectively (see figures 40a-40c). The normality assumption holds if the $\bar{W}$ has reached the steady state. At the begining of the transient state it is not possible to characterize its distribution, while when $\bar{W}$ is in a transient state we can see that its distribution takes an exponential or a X^2 form. This is left little by little as $\bar{W}$ comes close to the steady state

and then eventually it takes the normal form. The model SC#25#95 is a representative example of that result. In that model, although $\overline{W}$ has not reached the steady state yet (N=10000) we can observe the values of skewness having as 3.5, 1.8, 1.6 1.5 and the values of kurtosis as 24.7, 7.2, 5.7, 4.3 for N=2500, 5000, 7500, 10000 respectivelly. That is the values start from a high level, then come down, they pass from the actual values of γ_1, γ_2 for an exponential model and then continue decreasing and hopefully in the steady state they should reach a value around '0' where the value of skewness and kurtosis for the normal distribution. The same assumptions are obtained analyzing the plots of that model also. That is for N=2500 the plots are not linear under both EXPLT and NORMPL, for N=5000 a straight line appears under EXPLT, for N=7500 and 10000 the plot leaves gradually the straight line under EXPLT and begins to give us a straight line under NORMPL.



```
CENTRAL TENDENCY              SPREAD                      HIGHER CENTRAL MOMENTS        DISTRIBUTION
MEAN        1.943540E-02      VARIANCE   6.983232E-07     M3         9.837009E-11       MINIMUM                          1.710000E-02
MEDIAN      1.940000E-02      STD DEV    8.356574E-04     M4         1.438649E-12       .10 QUANTILE                     1.840000E-02
TRIMEAN     1.944555E-02      COEF VAR   4.286634E-02     SKEWNESS   1.685654E-01       .25 QUANTILE  (HINGE)            1.890000E-02
MIDMEAN     1.945880E-02      MEAN DEV   6.662000E-04     KURTOSIS  -4.986572E-02       .50 QUANTILE  (MEDIAN)           1.940000E-02
MIDRANGE    1.975000E-02      RANGE      5.300000E-03     BETA1      9.778069E-11       .75 QUANTILE  (HINGE)            2.010000E-02
GEOM MEAN   1.946756E-02      MIDSPREAD  1.199998E-03     BETA2      1.432974E-12       .90 QUANTILE                     2.060000E-02
HARM MEAN   1.944476E-02                                                                MAXIMUM                          2.240000E-02
```

Figure 40a - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. HISTOGRAM OF THE CUMULATED AND AVERAGED WAITING TIMES $\overline{W}_{10000}$ FROM THE SC#25#25; m=500 REPLICATIONS, RX=1.5; RS=6 .

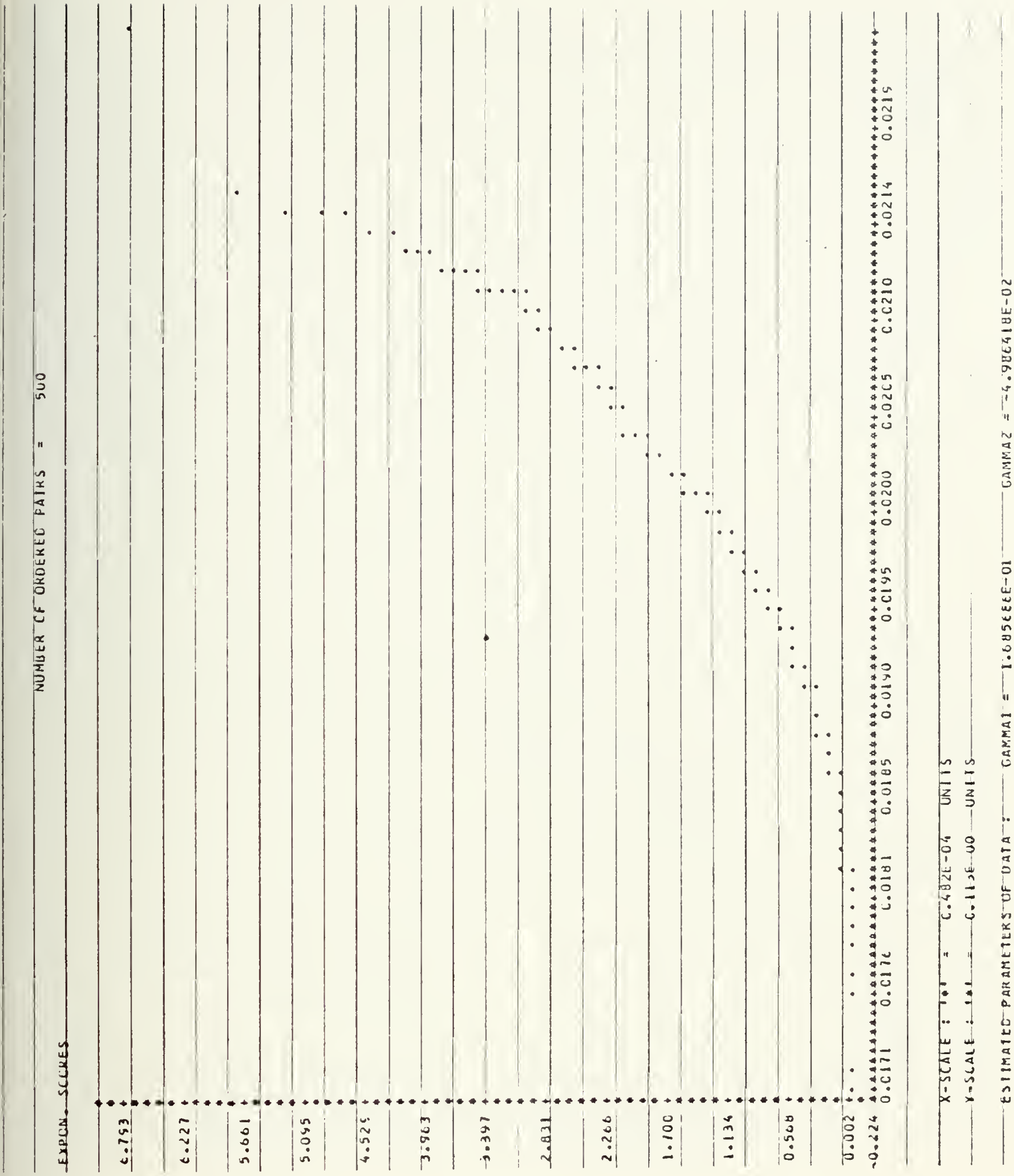


Figure 40b - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. EXPONENTIAL PLOT (EXPIT) OF THE CUMULATED AND AVERAGED WAITING TIMES $\bar{W}_{10000}$ FROM THE SC#25#25; m=500 REPLICATIONS, RX=1.5; RS=6 .

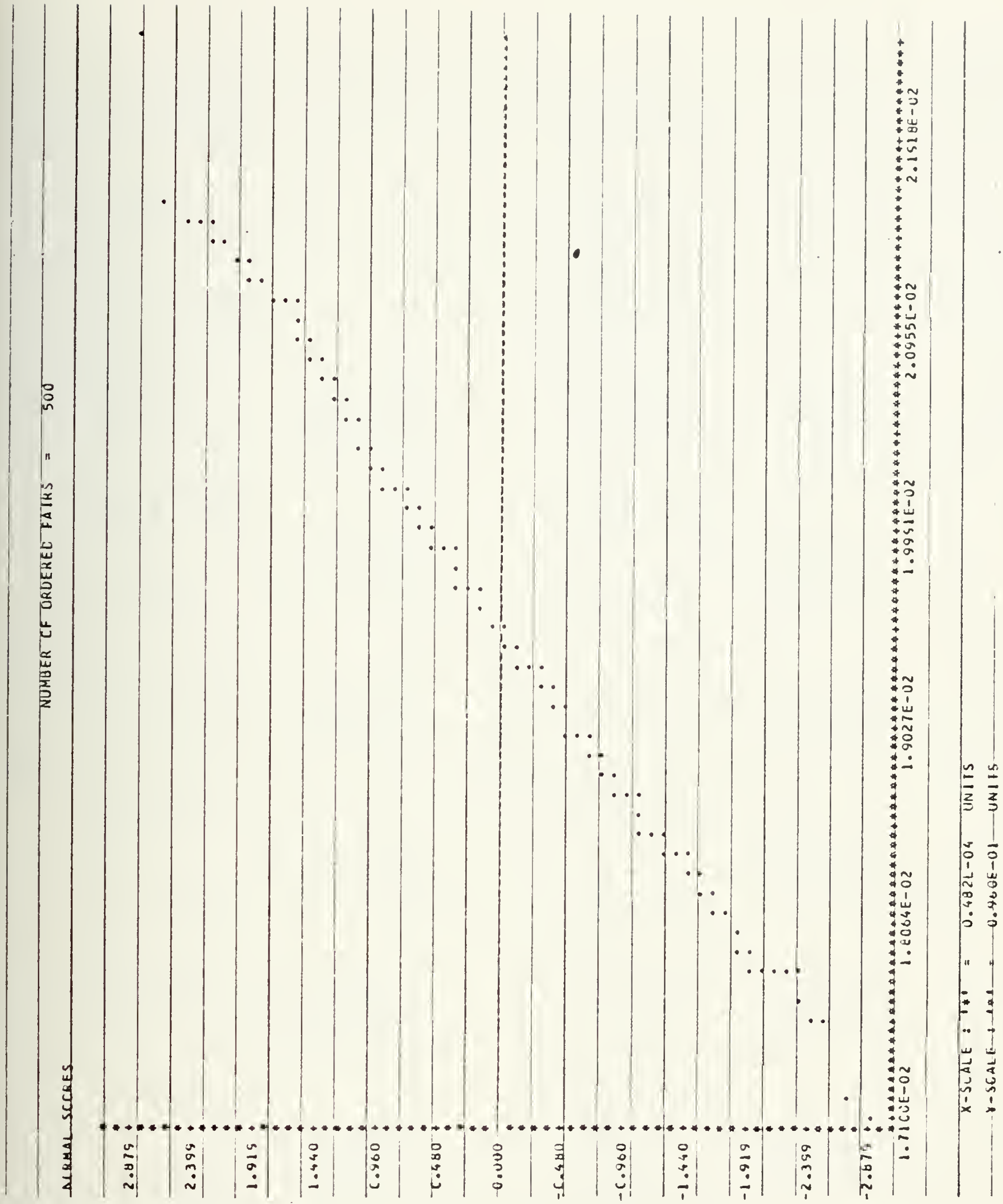


Figure 40c - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. NORMAL PLOT (NORMPL) OF THE CUMULATED AND AVERAGED WAITING TIMES $\bar{W}_{10000}$ FROM THE SC#25#25; m=500 REPLICATIONS, RX=1.5; RS=6 .

CORRELATED QUEUE

PARMTR	X 2	X 4	S 2	S 4	W 1 + CS	W 1	W 2 + CS	W 2	W 3 + OS	W 3	W 4 + CS	W 4
MEAN	3332.623	6659.074	832.243	1665.285	0.01722	0.10012	0.01623	0.09329	0.02002	0.12210	0.02410	0.11930
S(MEAN)	2.0640	2.8922	0.5507	0.7790	0.00225	0.00869	0.00230	0.00567	0.00309	0.01435	0.00324	0.01209
ST. DEV.	46.1521	64.6718	12.3133	17.4179	0.05035	0.08054	0.05153	0.09019	0.06920	0.12594	0.07250	0.12147
SKEWNESS	-0.0260	-0.2054	0.0729	-0.1521	3.51685	0.90211	4.25425	1.45247	5.30124	2.13677	5.26932	2.01606
KURTOS.	0.1763	0.1020	0.1254	-0.1511	13.19255	0.36415	20.11548	1.52827	33.07988	4.33559	40.42847	11.99954
ZEROS #	0.0	0.0	0.0	0.0	0.0	414.00000	0.0	413.00000	0.0	418.00000	0.0	399.00000

PARMTR	W 1 BAR	W 2 BAR	W 3 BAR	W 4 BAR	D 2	D 4
MEAN	0.0194362	0.0194240	0.0194738	0.0194854	-0.0001157	0.0006311
S(MEAN)	0.0000745	0.0000511	0.0000425	0.0000374	0.0003709	0.0002619
ST. DEV.	0.0016660	0.0011417	0.0009492	0.0008357	0.0082934	0.0058566
SKEWNESS	0.3750652	0.2163927	0.1402822	0.1685693	0.1290258	0.1707969
KURTOS.	0.1342526	-0.0230693	-0.2473210	-0.0458657	0.3181934	0.0662041
ZEROS #	0.0	0.0	0.0	0.0	0.0	0.0

UNCORRELATED QUEUE

PARMTR	X 2	X 4	S 2	S 4	W 1 + CS	W 1	W 2 + OS	W 2	W 3 + OS	W 3	W 4 + OS	W 4
MEAN	3332.823	6659.074	832.519	1665.033	0.04612	0.18748	0.04395	0.19976	0.06777	0.27550	0.06688	0.24408
S(MEAN)	2.0640	2.8922	0.5261	0.7704	0.00508	0.01455	0.00550	0.01854	0.00888	0.02904	0.00720	0.01941
ST. DEV.	46.1521	64.6718	11.7639	17.2262	0.11359	0.16141	0.12295	0.19440	0.19866	0.32205	0.16106	0.22719
SKEWNESS	-0.0260	-0.2054	0.2371	0.0944	3.11777	1.21373	4.00203	1.76568	4.78119	2.43433	3.59293	1.55721
KURTOS.	0.1783	0.1020	0.0274	0.0292	10.78564	1.41545	19.61831	3.32116	29.88885	7.72531	17.82196	6.29490
ZEROS #	0.0	0.0	0.0	0.0	0.0	377.00000	0.0	350.00000	0.0	377.00000	0.0	363.00000

PARMTR	W 1 BAR	W 2 BAR	W 3 BAR	W 4 BAR	D 2	D 4
MEAN	0.0554256	0.0553560	0.0554562	0.0554428	-0.0000607	0.0005961
S(MEAN)	0.0002257	0.0001594	0.0001390	0.0001222	0.0004250	0.0003002
ST. DEV.	0.0051357	0.0035648	0.0031081	0.0027328	0.0095036	0.0067129
SKEWNESS	0.4125256	0.3556815	0.2726265	0.3175142	0.1141328	0.2426832
KURTOS.	0.4375257	0.0793572	-0.0206966	-0.0520191	0.0510197	-0.0220528
ZEROS #	0.0	0.0	0.0	0.0	0.0	0.0

Figure 40d - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. TABULATION OF SAMPLE STATISTICS FOR THE DISTRIBUTIONS OF CUMULATED INTERARRIVAL TIMES AT N=5000 AND N=10000 (X 2 AND X 4), CUMULATED SERVICE TIMES (S 2 AND S 4), WAITING TIMES, WITH OR WITHOUT ZEROS (WI + 0, WI, I FOR CASE N=2500, 5000, 7500, 10000), CUMULATED WAITING TIMES (W1 BAR etc.) AND THE AVERAGED DIFFERENCES BETWEEN S2 AND X2 (D2); OBTAINED FROM m=500 REPLICATIONS OF THE RUN SC#25#25; EX=1.5; ES=6 .

4. The following values for $\bar{W}$ have been obtained from the runs with various values of t and r and for N=10000

N	t	r	$E[\bar{W}]$	$E[\bar{W}](M/M/1)$
10000	.25	.25	.1169	.3332
10000	.25	.50	.1724	.3332
10000	.25	.90	.3538	.3332
10000	.25	.95	.4075	.3332
10000	.25	.98	.4679	.3332
10000	.50	.25	.3831	.9985
10000	.50	.50	.5648	.9985
10000	.50	.90	1.4589	.9985
10000	.50	.95	2.1109	.9985
10000	.50	.98	3.7196*	.9985
10000	.95	.25	10.4672	18.3528
10000	.95	.50	13.4784*	18.3528
10000	.95	.90	29.1312*	18.3528
10000	.95	.95	75.1180*	18.3528
10000	.95	.98	145.3484*	18.3528
10000	.99	.25	35.5986*	53.3371
10000	.99	.50	43.1616*	53.3371
10000	.99	.90	98.9556*	53.3371
10000	.99	.95	139.0167*	53.3371
10000	.99	.98	219.7920*	53.3371

Analyzing the above results we may state conclusions as follows:

(i). For a given traffic intensity t the value of $\bar{W}$ is not constant, but depends on the value of correlation r, as happens in the queue with dependent service. We observe also that $\bar{W}$ increases as r increases, but the rate of increment is larger as r goes to 1, and very small for low

values of r; therefore $\bar{W}$ is not a linear function of r. Figure 41 gives the plots of $\bar{W}$ versus r for $t = .25$, $.50$, $.95$, $.98$ and also for the M/M/1 queue ($r=0$). Furthermore we can observe that the value of $\bar{W}$ for the correlated queue, comparing it with the result for the M/M/1 queue, is not always greater as in the case of dependent service, but depends on the value of r. It is less for $r \leq .50$ and greater for $r \geq .90$

(ii). For a given correlation the value of $\bar{W}$ increases as t increases. Furthermore high values of t cause high rate of increment, which becomes larger as the correlation increases. Figure 42 gives the plots of $\bar{W}$ versus t for $r = 0$, $.25$, $.50$, $.90$, $.95$, and $.98$. Observing that figure we can see also that for a given t the order of increment of $\bar{W}$ is for $r = .25$, $.50$, 0, $.90$, $.95$, $.98$ respectively. Thus we can conclude that the waiting time for the cross-correlated queue is generally greater than the waiting time for M/M/1 queue when the correlation is high.

5. The last result obtained from the simulation of that model was the convergence of W and $\bar{W}$. (These statistics also converge in value and in distribution slower than the same statistics in M/M/1 queue). The model SC#25#98 reveals that $\bar{W}$ converges faster than W (see figure 43). But the model SC#99#50 gave us the conclusion that W converges faster than $\bar{W}$ (see figure 44). Thus we can state here that for low values of t $\bar{W}$ converges faster than W and for high values of t the opposite is true.

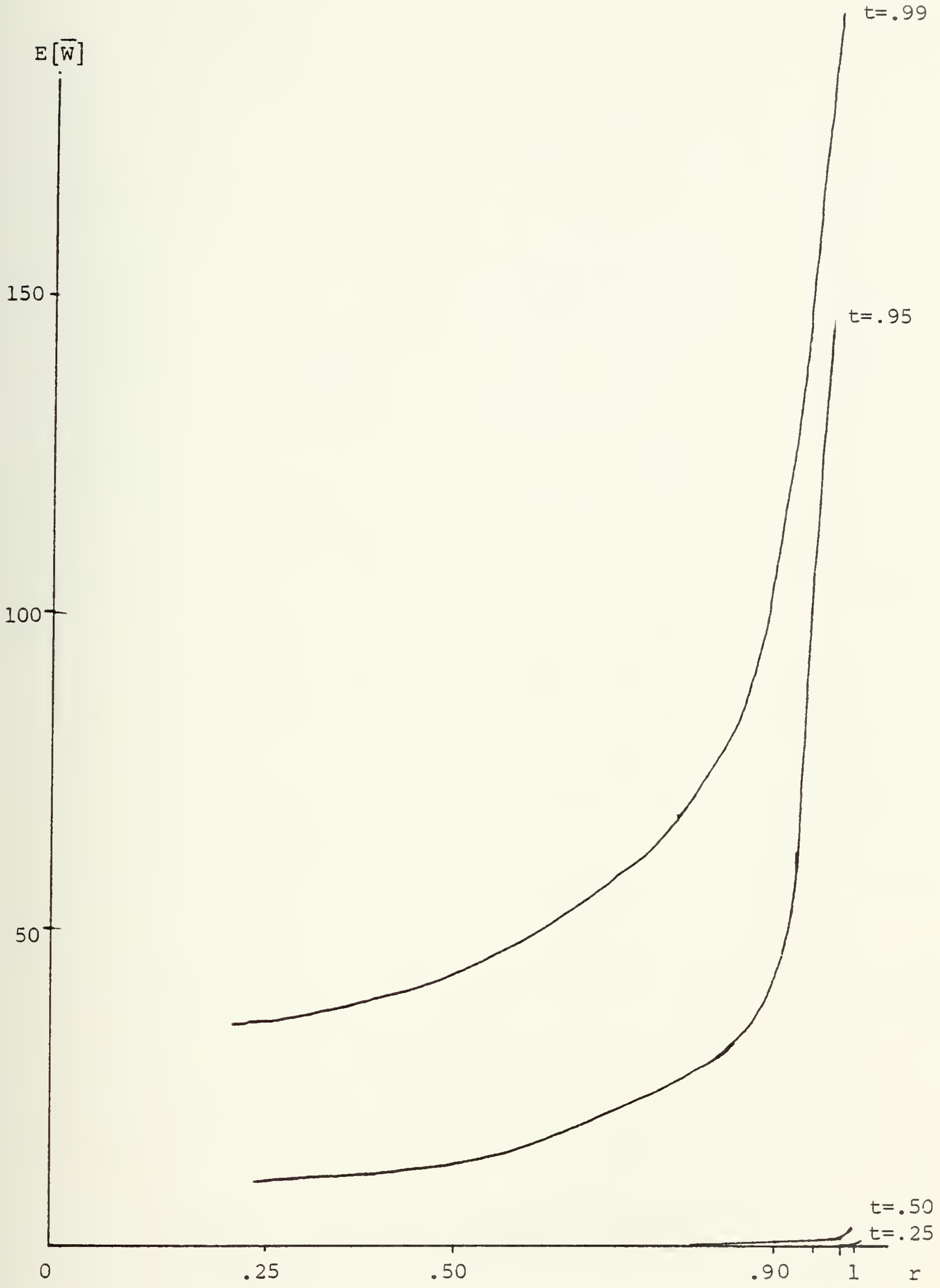


Figure 41 - QUEUE WITH CROSS-CORRELATED SERVICE TIMES AND POISSON INPUT. PLOT OF E[$\overline{W}$] VERSUS r.

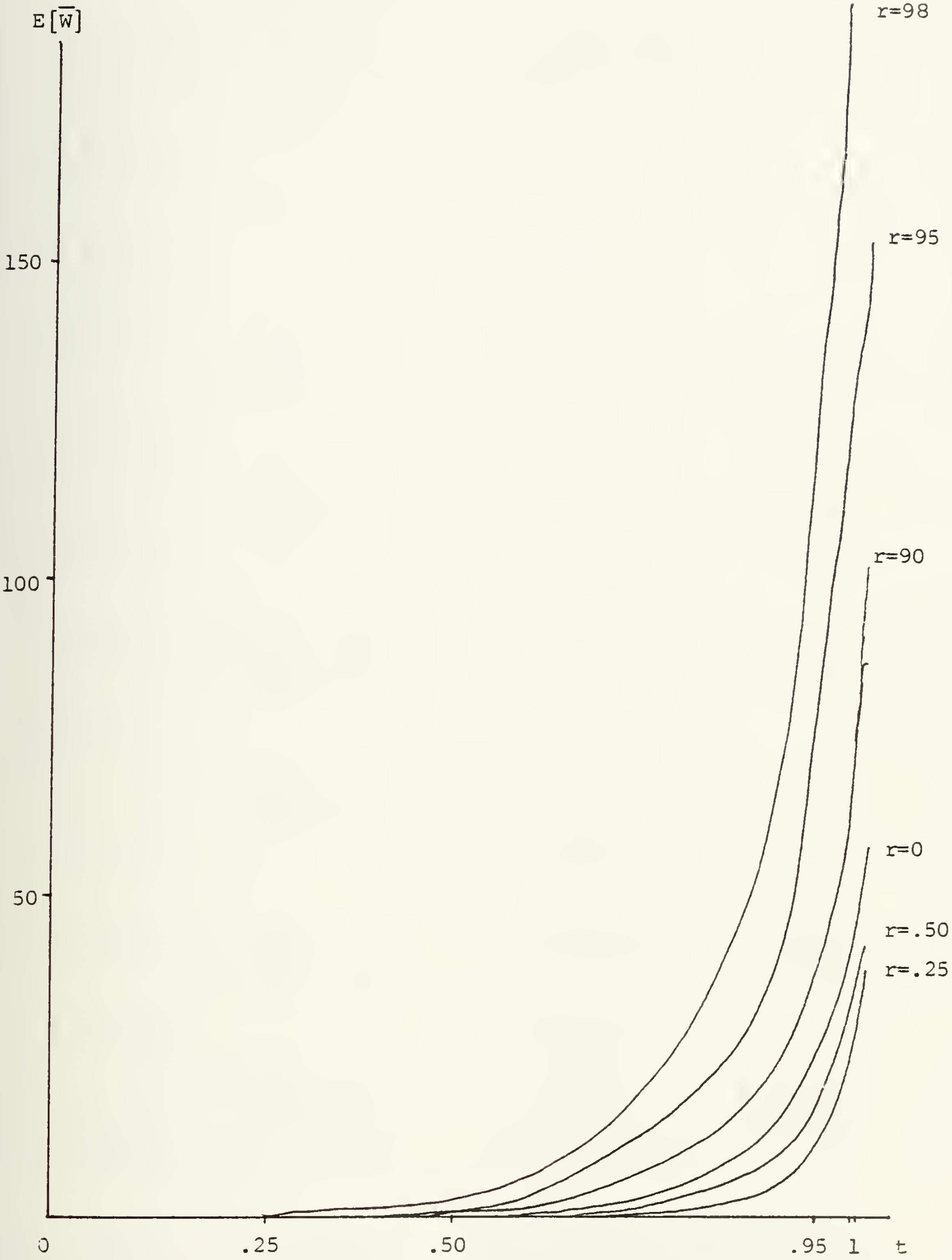


Figure 42 - QUEUE WITH CROSS-CORRELATED SERVICE TIMES AND POISSON INPUT. PLOT OF E[$\overline{W}$] VERSUS t.

```
                                              CORRELATED QUEUE

PARMTR        X 2        X 4        S 2        S 4     W 1 + OS      W 1     W 2 + OS      W 2     W 3 + OS      W 3     W 4 + OS      W 4

MEAN       3332.823   6659.074   835.466   1667.608    0.07436    0.29745    0.06703    0.27473    0.05545    0.25671    0.07050    0.31422
S(MEAN)       2.0640     2.8522    2.5856     3.8127    0.01383    0.05042    0.01071    0.03831    0.00833    0.03164    0.01259    0.03782
ST. DEV.     46.1521    64.6718   57.8163    65.2540    0.30920    0.56376    0.23952    0.42316    0.16617    0.33088    0.28163    0.45384
SKEWNESS     -0.0260    -0.2054    0.2924     0.2197    9.71833    5.26761    9.66360    5.25838    5.52440    2.58360    6.18225    3.60450
KURTOS.       0.1783     0.1020    0.1094     0.0155  118.93422   32.78246  118.62845   37.47824   38.10585    7.81849   51.61862   16.94318
ZEROS #       0.0        0.0       0.0        0.0        0.0     375.00000    0.0     378.00000    0.0     352.00000    0.0     356.00000

PARMTR        W 1  BAR        W 2  BAR        W 3  BAR        W 4  BAR          D 2             D 4

MEAN          0.0812984       0.0783889       0.0789859       0.0779790       0.0005325       0.0008533
S(MEAN)       0.0042510       0.0022484       0.0016570       0.0013266       0.0006404       0.0004425
ST. DEV.      0.0950555       0.0502757       0.0370524       0.0296679       0.0143202       0.0098544
SKEWNESS     14.1116364      11.5986595       8.8927116       7.3945675       0.0089558       0.0501304
KURTOS.     245.4133146     187.1348114     129.2479706      96.4026337      -0.1295548      -0.1657343
ZEROS #       0.0             0.0             0.0             0.0             0.0             0.0

                                             UNCORRELATED QUEUE

PARMTR        X 2        X 4        S 2        S 4     W 1 + OS      W 1     W 2 + OS      W 2     W 3 + OS      W 3     W 4 + OS      W 4

MEAN       3332.823   6659.074   832.519   1665.035    0.04612    0.18748    0.04395    0.19976    0.06777    0.27550    0.05688    0.24408
S(MEAN)       2.0640     2.8922    0.5261     0.7704    0.00508    0.01455    0.00550    0.01854    0.00888    0.02904    0.00720    0.01941
ST. DEV.     46.1521    64.6718   11.7639    17.2262    0.11359    0.16141    0.12259    0.19440    0.19866    0.32205    0.15106    0.22719
SKEWNESS     -0.0260    -0.2054    0.2371     0.0944    3.11777    1.21373    4.00203    1.76568    4.78119    2.43433    3.59293    1.95721
KURTOS.       0.1783     0.1020    0.0274     0.0292   10.78564    1.41949   19.01831    3.32118   29.88885    7.72531   17.82196    6.29490
ZEROS #       0.0        0.0       0.0        0.0        0.0     377.00000    0.0     390.00000    0.0     377.00000    0.0     363.00000

PARMTR        W 1  BAR        W 2  BAR        W 3  BAR        W 4  BAR          D 2             D 4

MEAN          0.0554256       0.0553960       0.0554962       0.0554428      -0.0000607       0.0005961
S(MEAN)       0.0002297       0.0001594       0.0001390       0.0001222       0.0004250       0.0003002
ST. DEV.      0.0051357       0.0035648       0.0031081       0.0027328       0.0095036       0.0067129
SKEWNESS      0.4125256       0.3556815       0.2726265       0.3175142       0.1141328       0.2426832
KURTOS.       0.4375257       0.0793972      -0.0206966      -0.0520191       0.0510197      -0.0220528
ZEROS #       0.0             0.0             0.0             0.0             0.0             0.0
```

Figure 43 - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. TABULATION OF SAMPLE STATISTICS FOR THE DISTRIBUTIONS OF CUMULATED INTERARRIVAL TIMES AT N=5000 AND N=10000 (X 2 AND X 4), CUMULATED SERVICE TIMES (S 2 AND S 4), WAITING TIMES, WITH OR WITHOUT ZEROS (WI + 0, WI, I FOR CASE N=2500, 5000, 7500, 10000), CUMULATED WAITING TIMES (W1 BAR etc.) AND THE AVERAGED DIFFERENCES BETWEEN S2 AND X2 (D2); OBTAINED FROM m=500 REPLICATIONS OF THE RUN SC#25#98; RX=1.5; RS=6 .

```
                                          CORRELATED QUEUE

PARMTR     X 2       X 4       S 2       S 4     W 1 + 0S    W 1     W 2 + 0S    W 2     W 3 + 0S    W 3     W 4 + 0S    W 4

MEAN     1683.007  3366.192  1664.470  3330.441  11.92533  12.14795  15.76974  16.02615  18.10533  18.32528  18.65758  18.92250
S(MEAN)     1.0424    1.4631    1.2327    1.7141   0.46255   0.46566   0.55157   0.55415   0.72168   0.72484   0.72730   0.73070
ST. DEV.   23.3082   32.7162   27.5631   38.3276  10.35185  10.31835  13.22781  13.17984  16.13724  16.11026  16.26302  16.22423
SKEWNESS   -0.0261   -0.2053    0.0879   -0.0557   1.17517   1.17321   1.28462   1.28536   1.20156   1.15862   1.26234   1.26211
KURTOS.     0.1783    0.1018    0.1017   -0.0342   1.32534   1.33178   1.84084   1.85598   1.10207   1.05401   1.43717   1.43475
SMPL SIZE 500.0000  500.0000  500.0000  500.0000 500.00000 491.00000 500.00000 452.00000 500.00000 454.00000 500.00000 493.00000

PARMTR       W 1  BAR      W 2  BAR      W 3  BAR      W 4  BAR        D 2          D 4

MEAN         8.6141525    11.2737417    13.2417622    14.3886738    -0.0003402   -0.0002082
S(MEAN)      0.2362964     0.2975569     0.3347992     0.3670427     0.0002464    0.0001764
ST. DEV.     5.2837515     6.6535178     7.4863405     8.2073265     0.0055160    0.0039438
SKEWNESS     1.6180201     1.6068602     1.6253405     1.5430393     0.0038780   -0.1288727
KURTOS.      3.3314552     3.0002041     3.1190050     2.6351595     0.5324717   -0.0878372
SMPL SIZE  500.0000000   500.0000000   500.0000000   500.0000000   500.0000000  500.0000000

                                         UNCORRELATED QUEUE

PARMTR     X 2       X 4       S 2       S 4     W 1 + 0S    W 1     W 2 + 0S    W 2     W 3 + 0S    W 3     W 4 + 0S    W 4

MEAN     1683.007  3366.192  1665.384  3331.023  14.82400  15.03446  18.82291  19.12897  21.96606  22.27755  23.66727  23.90634
S(MEAN)     1.0424    1.4631    1.0524    1.5408   0.56681   0.56925   0.75766   0.76221   0.91403   0.91937   1.02615   1.03093
ST. DEV.   23.3082   32.7162   23.5316   34.4538  12.67426  12.63930  16.94151  16.90674  20.43829  20.41335  22.94553  22.93684
SKEWNESS   -0.0261   -0.2053    0.2369    0.0945   1.14059   1.13946   1.44110   1.44187   1.34763   1.34367   1.90339   1.90468
KURTOS.     0.1783    0.1018    0.0275    0.0294   0.79820   0.79072   2.37382   2.37581   1.81030   1.80023   5.34098   5.34691
SMPL SIZE 500.0000  500.0000  500.0000  500.0000 500.00000 493.00000 500.00000 492.00000 500.00000 493.00000 500.00000 495.00000

PARMTR       W 1  BAR      W 2  BAR      W 3  BAR      W 4  BAR        D 2          D 4

MEAN        10.3916266    13.8082275    16.1548462    17.7791595    -0.0001574   -0.0001499
S(MEAN)      0.2797736     0.3830756     0.4490776     0.5010985     0.0002548    0.0002138
ST. DEV.     6.2559295     8.5658340    10.0416851    11.2049074     0.0065921    0.0047798
SKEWNESS     1.4325465     1.4735689     1.6260328     1.7394295     0.2896590    0.2765061
KURTOS.      1.9141417     2.0772343     2.9112906     3.5989595    -0.1199488    0.1787348
SMPL SIZE  500.0000000   500.0000000   500.0000000   500.0000000   500.0000000  500.0000000
```

Figure 44 - QUEUE WITH CROSS-CORRELATED SERVICE TIME SEQUENCE AND POISSON INPUT. TABULATION OF SAMPLE STATISTICS FOR THE DISTRIBUTIONS OF CUMULATED INTERARRIVAL TIMES AT N=5000 AND N=10000 (X 2 AND X 4), CUMULATED SERVICE TIMES (S 2 AND S 4), WAITING TIMES, WITH OR WITHOUT ZEROS (WI + 0, WI, I FOR CASE N=2500, 5000, 7500, 10000), CUMULATED WAITING TIMES (W1 BAR etc.) AND THE AVERAGED DIFFERENCES BETWEEN S2 AND X2 (D2): OBTAINED FROM m=500 REPLICATIONS OF THE RUN SC#99#50; EX=2.97; ES=3 .

```
C-----------------------------------------------------------------------
C
C
C     SUBROUTINE  NORMPL
C
C
C     PURPOSE
C
C         SUBROUTINE  NCRMPL  IS  INTENCED  TO :
C
C      1. PLOT  A SET  OF  DATA-PCINTS VS  EITHER NORMAL SCORES  OR
C                                                             -1
C     INVERSE  OF  STANDARD  NORMAL  DISTRIBUTION  FUNCTION  ( F   ).
C
C      2. COMPUTE THE VALUE OF  WILK-SHAPIRO TEST STATISTIC  ( W )  FOR
C     NCRMALITY.
C
C
C     CALLING SEQUENCES
C
C         CALL   NORMPL  ( X, SCORES, N ).
C
C
C     ARGUMENTS
C
C        X      A SINGLE DIMENSIONAL ARRAY OF DATA (REAL*4) TO BE PLCTTED
C               DIMENSIONED BY N .
C
C     SCORES    A CUMMY SINGLE DIMENSICNAL ARRAY (REAL*4) TO BE USED
C               FOR STORING NORMAL SCORES. DIMENSICNED EXACTLY AS X ARRAY
C
C        N      NUMBER OF DATA
C
C        K      AN  INTEGER  VALUE  FRCM  1  TO  3.
C               FCR :     K = 1   A PLCT IS ONLY GIVFN
C                         K = 2   THE  'W'-VALUE  IS  CNLY  GIVEN
C                         K = 3   BOTH  OF  THE  ABOVE  ARE  GIVEN.
C
C
C     USAGE
C
C         THE DATA WILL BE SORTED AND A SFT OF CRCER STATISTICS IS
C     PRODUCED.  THE ORDER STATISTICS TFEN ARE SCALED ON THE HORIZONTAL
C                                                                   -1
C     X-AXIS. ON THE OTHER HAND THE NORMAL SCORES CR THE CCMPUTED  F
C     ARE SCALED CN THE VERTICAL Y-AXIS.
C
C         IF THE NUMBER (N) OF DATA IS LESS THAN CR EQUALS TO 50 THEN
```

```
C                                                            -1
C     THE NORMAL SCORES (FOR  Y-AXIS) ARE USED, O.W. THE  F
C     IS COMPUTED AND IS USED.
C
C         THE NORMAL SCORES HAVE BEEN TAKEN FROM 'BIOMETRIKA TABLES
C     FOR STATISTICIANS'  VOL. I, THIRD EDITION P.190 AND ARE GIVEN BY
C     DATA STATEMENT IN THE SUBROUTINE.
C
C               -1
C         THE  F   VALUES ARE COMPUTED BY CALLING THE PROGRAM FUNCTION
C                                              -1
C     'INVNRM' WHICH EVALUATES THE VALUE OF   F  ( F (X    ) )
C                                                      (I)
C     WHERE  F (X    ) IS THE EMPIRICAL CUMULATIVE DISTRIBUTION FUNCTION
C                (I)
C     OF DATA GIVEN BY:  F (X    ) = I/(N+1) ,   I= 1, 2,...,N
C                            (I)
C
C         THE PROGRAM SUBROUTINE 'PLOT' IS THEN CALLED TO PLOT THE PAIRS
C                                             -1
C     OF ORDER STATISTICS-NORMAL SCORES/OR  F   .
C
C
C         IF  K=2  OR  K=3  THEN THE  PROGRAM  CALLS ITS SUBROUTINE
C     'WILKH' IN ORDER  TO  COMPUTE  THE  VALUE OF  'W'.
C
C         IF DATA ARE NORMAL A LINEAR FIT IS EXPECTED.
C
C
C     SUBROUTINES REQUIRED.
C
C         SUBROUTINE PXSORT IS USED AND IT HAS BEEN COMPILED AND ADDED
C     TO MPSLIB.
C
C
C     PROGRAMMER:  GEORGIOS J. DANIKAS
C
C                  UNDER DIRECTIONS OF PROF.  P. A. W. LEWIS
C
C     DATE:  APR 77
C
C-----------------------------------------------------------------------
C
C
C
```

```
SUBROUTINE   NORMPL  ( X, SCORES, N, K )

REAL *8  INVNRM, W
LOGICAL *1  ODD
DIMENSION  X(N), SCORES(N), BIOMET(403),
*           SCORE1(200), SCORE2(203)
DATA   SCORE1
*      / .564, .846, 1.029, .297, 1.163, .495, 1.267, .642, .202,
*       1.352, .757, .353, 1.424, .852, .473,.153, 1.485, .932,.572,
*        .275, 1.539, 1.001, .656, .376, .123, 1.586, 1.062, .729,
*       .462, .225, 1.629, 1.116, .793, .537, .312, .103,1.668,
*      1.164, .850, .603, .388, .19, 1.703, 1.208, .901, .662, .456,
*       .267, .088, 1.736, 1.248, .948, .715, .516, .335, .165,
*      1.766, 1.285, .99, .763, .57, .396, .234, .077, 1.794,
*      1.319, 1.029, .807, .619, .451, .295, .146, 1.82, 1.35,
*      1.066, .848, .665, .502, .351, .208, .069, 1.844, 1.38, 1.099
*     ,.886, .707, .548, .402, .264, .131, 1.867, 1.408, 1.131, .921
*     ,.745, .59, .448, .315, .187, .062, 1.89, 1.43, 1.16, .95, .78
*     ,.63, .49, .36, .24, .12, 1.91, 1.46, 1.19, .98, .82, .67, .53
*     ,.41, .29, .17, .06, 1.93, 1.48, 1.21, 1.01, .85, .7, .57, .45
*     ,.33, .22, .11, 1.95, 1.5, 1.24, 1.04, .88, .73, .6, .48,
*      .37, .26, .16, .05, 1.97, 1.52, 1.26, 1.07, .91, .76 ,
*      .64,.52, .41, .3, .2, .1, 1.98, 1.54, 1.29, 1.09,.93,.79,.67,
*      .55, .44, .34, .24, .14, .05, 2.01, 1.58, 1.33, 1.14, .98,
*      .85, .73, .61, .51, .41, .32, .22, .13, .04, 2.04, 1.62, 1.36
*     ,1.18,1.03,.89,.78,.67,.57,.47,.38,.29,.21,.12,.04,2.07,1.65/
DATA    SCORE2
*       /  1.4, 1.22, 1.07, .94, .82, .72, .62, .53, .44,
*       .36, .28, .2, .12, .04, 2.09, 1.68, 1.43, 1.25, 1.11, .98,
*       .87, .76, .67, .58, .5, .41, .34, .26, .18, .11, .04, 2.12,
*      1.7, 1.46, 1.28, 1.14, 1.02, .91, .81, .71, .63, .54, .47,
*       .39, .32, .24, .17, .1, .03, 2.14, 1.73, 1.49, 1.32, 1.17,
*     1.05, .94, .85, .75, .67, .59, .51, .44, .37, .3, .23, .16,
*       .1, .03, 2.16, 1.75, 1.52, 1.34, 1.2, 1.08, .98, .88, .79,
*       .71, .63, .56, .49, .42, .35, .28, .22, .16, .09, .03  ,
*     2.18, 1.78, 1.54, 1.37, 1.23, 1.11, 1.01, .91, .83,.75,.67,.6,
*       .53, .46, .4, .33, .27, .21, .15, .09, .03, 2.2, 1.8, 1.57,
*      1.4, 1.26, 1.14, 1.04, .95, .86, .78, .71, .64, .57, .5,
*       .44, .38, .32, .26, .2, .14, .09, .03, 2.22, 1.82, 1.59,
*      1.42, 1.28, 1.17, 1.07, .98, .89, .81, .74, .67, .6, .54,
*       .48, .42, .36, .3, .25, .19, .14, .08, .03, 2.23, 1.84,
*      1.61, 1.44, 1.31, 1.19, 1.09, 1., .92, .84, .77, .7, .64,
*       .58, .52, .46, .4, .34, .29, .24, .18, .13, .08, .03, 2.25,
*      1.85, 1.63, 1.46, 1.33, 1.22, 1.12, 1.03, .95, .87, .8, .74,
*       .67, .61, .55, .49, .44, .38, .33, .28, .23, .18, .13,
*       .07, .03 /
EQUIVALENCE  ( BIOMET(1),SCORE1(1) ), ( BIOMET(201),SCORE2(1) )
ODD = .TRUE.
```

```
      IF  ( N .LE. 1 )    GO TO 35
      CALL   PXSORT  ( X, 1, N )
      NHALF = N / 2
      GC TO (15, 25, 15), K
   25 CALL   WILK   ( X, N, NHALF, W, SCCRES )
      RETURN
   15 IF  ( MOD(N,2) .EQ. 0 )    ODD = .FALSE.
      IF  ( N .GT. 50 )    GO TO 40
      INDEX = (  (N-1)**2 + 3 ) / 4 - 1
      IF  ( ODD  .AND.   N .LE. 25 )    INDEX = (N-1)**2 / 4
      IF  ( .NOT. ODD   .AND.   N .GE. 26 )   INDEX = (N-26)*(N+24)/8+156
      IF  ( ODD  .AND.   N .GE. 26 )   GC TO 20
C
      DC 11   I = 1,NHALF
         SCORES(N-I+1) = BICMET(INDEX+I)
   11    SCORES(I) = -BIOMET(INDEX+I)
      GC TO 30
   20 INDEX = (N-27)*(N+23)/8 + 156
      DC 22   I = 1,NHALF
         SCORES(N-I+1) = ( BIOMET(INDEX+I) + BIOMFT(INDEX+I+NHALF) ) / 2
   22    SCORES(I) = -SCORES(N-I+1)
   30 IF  ( ODD )    SCORES(NHALF+1) = 0.
      CALL   PLOT   ( X, SCORES, N )
      IF  ( K .EQ. 3 )    CALL    WILK   ( X, N, NHALF, W, SCCRES )
      RETURN
C
   40 NHALF = N / 2
      DC 44   I = 1,NHALF
         SCORES(N-I+1) = INVNRM ( DFLOAT(N-I+1) / (N+1) )
   44    SCORES(I) = -SCORES(N-I+1)
      GC TO 30
   35 WRITE (6,200)
  200 FCRMAT  ( '1', 5X, '*** NUMBER OF DATA TO SMALL ***' )
      RETURN
      END


      REAL  FUNCTION   INVNRM *8 ( PR )


      REAL *8  PR
         IF  ( PR .LT. 0.5D0 )   PR = 1.-PR
         PR  =  DSQRT ( DLOG(1.0D0/(1.0D0-PR)**2) )
      INVNRM   =    PR   -  ( ( .010328*PR + .802853)*PR + 2.515517 ) /
     *   ( ( (.001308*PR+.189269)*PR + 1.432788)*PR + 1. )
      RETURN
      END
```

```
      SLBROUTINE   PLOT  ( X, SCORES, N )

      DIMENSION   X(N), SCORES(N), XAXIS(12)
      LOGICAL *1  GRAPH(65,112), FMTF(16), DIGITS(7)
      LOGICAL *1  VERT, BAR, STAR, BLANK, CROSS, DOT, SAMEX
      DATA        VERT, BAR, STAR, BLANK, CROSS, DCT
     *            / '|', '-', '*', ' ', '+', '.'  /
      DATA  DIGITS  / '1','2','3','4','5','6','7' /
     *      FMTF    / '(','4','X',',','1','1','(','F','7','.','1',',',
     *                '3','X',')',')'  /
      SAMEX = .TRUE.
C
      DO 11  I = 1,65
         DO 11  J = 1,112
   11       GRAPH(I,J) = BLANK
C
      DC 22  I = 1,65
         GRAPH(I,1) = STAR
         IF  ( MOC(I,5) .EQ. 3 )   GRAPH(I,1) = CROSS
   22 CONTINUE
C
      DC 33  J = 1,112
         GRAPH(65,J) = STAR
         IF  ( MOD(J,10) .EQ. 2 )   GRAPH(65,J) = CROSS
   33    GRAPH(33,J) = BAR
C
      IF  ( X(1)*X(N) .LT. 0. )   SAMEX = .FALSE.
      XMAX = AMAX1 ( ABS(X(1)), ABS(X(N)) )
      XRANGE = X(N) - X(1)
      YRANGE = 2*SCORES(N)
      XDELTA = XRANGE / 110
      YDELTA = YRANGE / 60
      IF  ( XRANGE .LE. 0. )   GO TO 45
      IF  ( SAMEX )   GO TO 10
      JZERO =  2 + INT( .5-110*X(1)/XRANGE )
C
      DO 44  I = 1,65
   44    GRAPH(I,JZERO) = VERT
C
   1C CONTINUE
      DC 66  K = 1,N
         I = INT ( .5 + 60 * (SCORES(N-K+1) - SCORES(1)) / YRANGE ) + 3
         J = INT ( .5 + 110* ( X(K)-X(1) ) / XRANGE ) + 2
         GRAPH(I,J) = DOT
   66 CONTINUE
```

```
      II = -3
      WRITE (6,201)   N
      DO 77  I = 1,65
         II = II+1
         IF  ( MOD(I,5) .NE. 3 )   GO TO 15
         YAXIS = SCORES(N) - YDELTA*II
         WRITE (6,200)   YAXIS,   ( GRAPH(I,J), J=1,112 )
         GO TO 77
   15    WRITE (6,400)   ( GRAPH(I,J), J=1,112 )
   77 CONTINUE
      XAXIS(1) = X(1)
      DO 88  J = 2,12
   88    XAXIS(J) = XAXIS(J-1) + 10*XDELTA
      IF  ( XMAX .LT. .001  .OR.  XMAX .GE. 1.E04  )  GO TO 25
      NDIG =  INT ( 2. + ABS (ALOG10(XMAX) )  )
      NFRCT =  INT ( 1. + ABS (ALOG10 ( XRANGE/11. ) )  )
      IF  ( XRANGE .GE. 11. )   NFRCT = 1
      NONF = NDIG+NFRCT
      IF  ( NFRCT .GT. 4  .OR.  NONF .GE. 7 )   GO TO 25
      FMTF(2) = DIGITS(NFRCT+3)
      FMTF(11) = DIGITS(NFRCT)
      WRITE (6,FMTF)   ( XAXIS(J), J=1,11 )
      GO TO 40
   25 WRITE (6,600)   ( XAXIS(J), J=1,11,2 )
   40 WRITE (6,604)  XDELTA, YDELTA
      RETURN
   45 WRITE (6,601)
      RETURN
C
  200 FORMAT  ( 1X, F6.3, 1X, 112A1 )
  201 FORMAT  ( '1' ,  50X,  'NUMBER OF ORDERED PAIRS  = ', I6 // 2X,
     *        'NORMAL SCORES ' //)
  400 FORMAT  (   8X, 112A1 )
  600 FORMAT  ( 4X, 6(1PE11.4, 9X ) )
  601 FORMAT  ( // 5X, '*** GIVEN DATA HAVE CONSTANT VALUE ***' )
  604 FORMAT  (////, 8X, 'X-SCALE : ''*''  =  ',E10.3, '   UNITS',
     *          //, 8X, 'Y-SCALE : ''*''  =  ', E10.3 , '   UNITS' )
      END


      SUBROUTINE   WILK  ( X, N, K, W, A )


      REAL *8        W, SUM, S2, B, C, INVNRM
      DIMENSION      A(N), WD(625), WD1(199), WD2(209), WD3(209), WD4(8),
     *               X(N)
      DATA           WD1 /
```

```
*.7071,.7071,.6872,.1677,.6646,.2413,.6431,.2806,.0875,.6233,.3031,
*.1401,.6052,.3164,.1743,.0561,.5888,.3244,.1976,.0947,.5739,.3291,
*.2141,.1224,.0399,.5601,.3315,.2260,.1429,.0695,.5475,.3325,.2347,
*.1586,.0922,.0303,.5359,.3325,.2412,.1707,.1099,.0539,.5251,.3318,
*.2460,.1802,.1240,.0727,.0240,.5150,.3306,.2495,.1878,.1353,.0880,
*.0433,.5056,.3290,.2521,.1939,.1447,.1005,.0593,.0196,.4968,.3273,
*.2540,.1988,.1524,.1109,.0725,.0359,.4886,.3253,.2553,.2027,.1587,
*.1197,.0837,.0496,.0163,.4808,.3232,.2561,.2059,.1641,.1271,.0932,
*.0612,.0303,.4734,.3211,.2565,.2085,.1686,.1334,.1013,.0711,.0422,
*.0140,.4643,.3185,.2578,.2119,.1736,.1399,.1092,.0804,.0530,.0263,
*.4590,.3156,.2571,.2131,.1764,.1443,.1150,.0878,.0618,.0368,.0122,
*.4542,.3126,.2563,.2139,.1787,.1480,.1201,.0941,.0696,.0459,.0228,
*.4493,.3098,.2554,.2145,.1807,.1512,.1245,.0997,.0764,.0539,.0321,
*.0107,
*.4450,.3069,.2543,.2148,.1822,.1539,.1283,.1046,.0823,.0610,.0403,
*.0200,.4407,.3043,.2533,.2151,.1836,.1563,.1316,.1089,.0876,.0672,
*.0476,.0284,.0094,.4366,.3018,.2522,.2152,.1848,.1584,.1346,.1128,
*.0923,.0728,.0540,.0358,.0178,.4328,.2992,.2510,.2151,.1857,.1601,
*.1372,.1162,.0965,.0778,.0598,.0424,.0253,.0084,.4291,.2968,.2499/
 DATA         WD2  /
*.2150,.1864,.1616,.1395,.1192,.1002,.0822,.0650,.0483,.0320,.0159,
*.4254,.2944,.2487,.2148,.1870,.1630,.1415,.1219,.1036,.0862,.0697,
*.0537,.0381,.0227,.0076,.4220,.2921,.2475,.2145,.1874,.1641,.1433,
*.1243,.1066,.0899,.0739,.0585,.0435,.0289,.0144,.4188,.2898,.2463,
*.2141,.1878,.1651,.1449,.1265,.1093,.0931,.0777,.0629,.0485,.0344,
*.0206,.0068,.3894,.4156,.2876,.2451,.2137,.1880,.1660,.1463,.1284,
*.1118,.0961,.0812,.0669,.0530,.0395,.0262,.0131,.4127,.2854,.2439,
*.2132,.1882,.1667,.1475,.1301,.1140,.0988,.0844,.0706,.0572,.0441,
*.0314,.0187,.0062,.4096,.2834,.2427,.2127,.1883,.1673,.1487,.1317,
*.1160,.1013,.0873,.0610,.0484,.0361,.0239,.0119,.4068,.2813,.2415,
*.2121,.1883,.1678,.1496,.1331,.1179,.1036,.0900,.0770,.0645,.0523,
*.0404,.0287,.0172,.0057,.4040,.2794,.2403,.2116,.1883,.1683,.1505,
*.1344,.1196,.1056,.0924,.0798,.0677,.0559,.0444,.0331,.0220,.0110,
*.4015,.2774,.2391,.2110,.1881,.1686,.1513,.1356,.1211,.1075,.0947,
*.0824,.0706,.0592,.0481,.0372,.0264,.0158,.0053,.3989,.2755,.2380,
*.2104,.1880,.1689,.1420,.1366,.1225,.1092,.0967,.0848,.0733,.0622,
*.0515,.0409,.0305,.0203,.0101,.3964,.2737,.2368,.2098,.1878,.1691,
*.1526,.1376,.1237,.1108,.0986,.0870,.0759,.0651,.0546,.0444,.0343,
*.0244,.0146,.0049,.3940,.2719,.2357,.2091,.1876,.1693,.1531,.1384/
 DATA         WD3  /
*.1249,.1123,.1004,.0891,.0782,.0677,.0575,.0476,.0379,.0283,.0188,
*.0094,.3917,.2701,.2345,.2085,.1874,.1694,.1535,.1392,.1259,.1136,
*.1020,.0909,.0804,.0701,.0602,.0506,.0411,.0318,.0227,.0136,.0045,
*.3894,.2684,.2334,.2078,.1871,.1695,.1539,.1398,.1269,.1149,.1035,
*.0927,.0824,.0724,.0628,.0534,.0442,.0352,.0263,.0175,.0087,.3872,
*.2667,.2323,.2072,.1868,.1695,.1542,.1405,.1278,.1160,.1049,.0943,
*.0842,.0745,.0651,.0560,.0471,.0383,.0296,.0211,.0126,.0042,.3850,
*.2651,.2313,.2065,.1865,.1695,.1545,.1410,.1286,.1170,.1062,.0959,
```

```
     *.0860,.0765,.0673,.0584,.0497,.0412,.0328,.0245,.0163,.0081,.3830,
     *.2635,.2302,.2058,.1862,.1695,.1548,.1415,.1293,.1180,.1073,.0972,
     *.0876,.0783,.0694,.0607,.0522,.0439,.0357,.0277,.0197,.0118,.0039,
     *.3808,.2620,.2291,.2052,.1859,.1695,.1550,.1420,.1300,.1189,.1085,
     *.0986,.0892,.0801,.0713,.0628,.0546,.0465,.0385,.0307,.0229,.0153,
     *.0076,.3789,.2604,.2281,.2045,.1855,.1693,.1551,.1423,.1306,.1197,
     *.1095,.0998,.0906,.0817,.0731,.0648,.0568,.0489,.0411,.0335,.0259,
     *.0185,.0111,.0037,.3770,.2589,.2271,.2038,.1851,.1692,.1553,.1427,
     *.1312,.1205,.1105,.1010,.0919,.0832,.0748,.0667,.0588,.0511,.0436,
     *.0361,.0288,.0215,.0143,.0071,.3751,.2574,.2260,.2032,.1847,.1691,
     *.1554,.1430,.1317,.1212,.1113,.1020,.0932,.0846,.0764,.0685,.0608/
      DATA          WD4 /
     *.0532,.0459,.0386,.0314,.0244,.0174,.0104,.0035 /
C
      EQUIVALENCE  ( WD(1),WD1(1) ), ( WD(200),WD2(1) ),
     *             ( WD(409),WD3(1) ), ( WD(618),WD4(1) )
      IF  ( N .LE. 50 )   GO TO 15
      C = -2.722D0 + N*4.083D0
      C = DSQRT(C)
      DO 11  I = 2,K
   11    A(I) = 2 * INVNRM (DFLOAT(N-I+1) / (N+1)) / C
      B = (N+1.D0) / 2.D0
      C = N/2.D0 + 1.D0
      A(1) = DGAMMA(B) / ( SQRT(2.)*DGAMMA(C) )
      A(1) = SQRT(A(1))
C
   15 SUM = 0.D0
      S2 = 0.D0
      B = 0.D0
      IA = K*(K-1) + K*MOD(N,2)
C
      DO 22  I = 1,N
         SUM = SUM + X(I)
         IF  ( I .LE. K  .AND.  N .LE. 50 )   A(I) = WD(IA+I)
   22 CONTINUE
C
      SUM = SUM/N
      DO 33  I = 1,N
         S2 = S2 + ( X(I) - SUM )**2
         IF  ( I .LE. K )   B = B + A(I)*( X(N-I+1) - X(I) )
   33 CONTINUE
      W = B**2/S2
      WRITE (6,200)   W
  200 FORMAT  ( //8X, 'WILK-SHAPIRO TEST:  TEST VALUE = ', 1PE10.3, 4X,
     *            '   SEE BIOMETRIKA (1965), V. 52, P. 591, FOR CRITICAL',
     *            ' VALUES' )
      RETURN
      END
```

```
C-----------------------------------------------------------------------
C
C
C     SUBROUTINE EXPLT
C
C
C
C     PURPOSE
C
C
C        1. SUBROUTINE EXPLT PLOTS A SET OF DATA VS EXPONENTIAL SCCRES
C
C        2. ESTIMATES AND GIVES THE PARAMETERS  GAMMA1, GAMMA2.
C
C
C     CALLING SEQUENCES
C
C        CALL   EXPLT  ( X, SCORES, N )
C
C
C     ARGUMENTS
C
C
C         X    A SINGLE DIMENTIONAL ARRAY (REAL*4) OF DATA TO BE PLOTTED
C              DIMENSIONED BY N.
C
C      SCORES  A DUMMY SINGLE DIMENSIONAL ARRAY (REAL*4) TO BE USED FOR
C              STORING EXPON. SCORES. DIMENSIONED EXACTLY AS X ARRAY IS.
C
C         N    NUMBER OF DATA  (MUST BE GREATER THAN 3)
C
C
C
C    USAGE
C
C
C       THE DATA WILL BE SORTED AND A SET OF ORDER STATISTICS IS GIVEN.
C    THE ORDER STATISTICS THEN ARE SCALED ON THE HORIZONTAL X-AXIS.
C    ON THE OTHER HAND THE EXPON. SCORES ARE EVALLATED AND SCALED ON
C    THE VERTICAL Y-AXIS.
C
C                                                    I
C       FOR EVALUATION OF EXPON. SCORES THE FORMULA SUM(1/(N-J+1)), I=1,
C                                                   J=1
C    2,...,N IS USED BASED ON THE FACT THAT 'THE EXPECTED VALUE OF THE
C     TH                           I
C    I   ORDER STATISTIC EQUALS SUM( 1/(N-J+1) ), J=1,2,...,I', IF THE
C                               J=1
```

```
C     DATA ARE REALLY EXPON. DISTRIBUTED.
C
C        THE PARAMETERS GAMMA1, GAMMA2 ARE EVALUATED USING THE FORMULAS:
C
C                           I                                     1.5
C        GAMMA1 = ( N* SUM(X    -X̄ )   /((N-1)*(N-2)) )/ VAR
C                      J=1  (I)
C
C                                      I
C        GAMMA2 = ( (N*(N-2)+3) SUM(X    -X̄) / ( (N-1)*N-2) )-
C                               J=1  (I)
C                        2                                          2
C              - 3*VAR (N-1)(2*N-3) / ( N*(N-2)*(N-3) )    ) /  VAR - 3
C
C        THE PROGRAM SUBROUTINE 'EPLOT' IS CALLED TO PLOT THE PAIRS OF
C     ORDER STATISTICS-EXPON. SCORES.
C
C        IF DATA ARE EXPONENTIALLY DISTRIBUTED A LINEAR FIT IS EXPECTED.
C
C
C     SUBROUTINES REQUIRED
C
C        SUBROUTINE 'PXSORT' IS USED AND IT HAS BEEN COMPILED AND ADDED
C     TO MPSLIB.
C
C
C
C     ERROR CONDITIONS
C
C         NO OUTPUT IS EXPECTED IF :
C
C         1.  X    IS LESS THAN ZERO  FOR EVERY  I=1,2,...,N.
C
C         2.  DATA HAVE CONSTANT VALUE.
C
C         3.  N IS LESS THAN 4 .
C
C
C     PROGRAMMER:  GEORGIOS J. DANIKAS
C
C                  UNDER DIRECTIONS OF PROF.  P. A. W. LEWIS
C
C     DATE:  APR 77
C
C
C-----------------------------------------------------------------------
C
C
```

```
      SLBROUTINE   EXPLT  ( X, SCORES, N )

      REAL *8  SUM1, SUM2, SUM3, SUM4, DIF, XM3, XM4
C
      DIMENSION  X(N), SCORES(N)
      IF  ( N .LE. 3 )  GO TO 35
      CALL   PXSORT  ( X, 1, N )
      IF  ( X(1) .LT. 0. )   GO TO 25
      IF  ( X(1) .EQ. X(N) )   GO TO 30
C
C     CCMPUTE THE EXPON. SCCRES.
C
      XN = N
      SUM1 = X(1)
      SCORES(1) = 1/XN
      DO 11  I = 2,N
         SUM1 = SUM1 + X(I)
 11      SCORES(I) = SCCRES(I-1) + 1./(XN-I+1)
C
      CALL   EPLOT  ( X, SCORES, N )
C
C     COMPUTE  GAMMA1 AND GAMMA2  PARAMETERS
C
      SUM2 = 0.DO
      SUM3 = 0.DO
      SUM4 = 0.DO
      XMEAN = SUM1/XN
C
      DC 22  I = 1,N
         DIF = X(I) - XMEAN
         SUM2 = SUM2 + DIF**2
         SUM3 = SUM3 + DIF**3
         SUM4 = SUM4 + DIF**4
 22   CONTINUE
      XVAR = SUM2 / (XN-1)
      XM3 = XN*SUM3 / ((XN-1)*(XN-2))
      XM4 = SUM4*(3+(XN-2)*XN) / (  (XN-1)*(XN-2)*(XN-3)  )
     *       -3*XVAR**2*(XN-1)*(2*XN-3) / ( XN*(XN-2)*(XN-3) )
      GAMMA1 = XM3 / XVAR**1.5
      GAMMA2 = XM4 / XVAR**2 - 3.
      WRITE (6,200)   GAMMA1, GAMMA2
      RETURN
 25   WRITE  (6,400)
      RETURN
 30   WRITE  (6,401)
      RETURN
 35   WRITE (6,402)
```

```
      RETURN
  2CC FORMAT   ( //8X, 'ESTIMATED PARAMETERS OF DATA :      GAMMA1 = ',
     *          1PE13.6, '        GAMMA2 = ',1PE13.6 )
  400 FCRMAT   ( // 5X, '*** DATA ARE NCT EXPONENTIALY DISTRIBUTED ***' )
  401 FORMAT   ( // 5X, '*** GIVEN DATA HAVE CONSTANT VALUE ***' )
  402 FCRMAT   ( //, 5X, '*** NUMBER OF CATA TOO SPALL ***' )
      END


      SLBROUTINE   EPLOT  ( X, SCORES, N )


      DIMENSION  X(N), SCORES(N), XAXIS(12)
      LOGICAL *1  GRAPH(65,112), FMTF(16), DIGITS(7)
      LCGICAL *1    STAR, CROSS, DOT, BLANK
      DATA          STAR, CROSS, DOT, BLANK
     *          /   '*',   '+',   '.',   ' '     /
      DATA  DIGITS  / '1','2','3','4','5','6','7' / ,
     *      FMTF    / '(','4','X',',','1','1','(','F','7','.','1',',',
     *                '3','X',')',')'   /
C
      DC 11   I = 1,65
         DO 22  J = 1,112
            GRAPH(I,J) = BLANK
            GRAPH(65,J) = STAR
            IF  ( I .EQ. 65  .AND.  MOD(J,10) .EQ. 2 ) GRAPH(I,J)=CROSS
   22    CONTINUE
         GRAPH(I,1) = STAR
         IF  ( MOD(I,5) .EQ. 3 )   GRAPH(I,1) = CROSS
   11 CONTINUE
C
      XRANGE = X(N) - X(1)
      YRANGE = SCORES(N)-SCORES(1)
      XDELTA = XRANGE / 110
      YDELTA = YRANGE / 60
      DC 66  K = 1,N
         I = INT (3.5 + 60 * (SCORES(N) - SCORES(K) ) / YRANGE )
         J = INT (2.5 + 11C* ( X(K)-X(1) ) / XRANGE )
         GRAPH(I,J) = DOT
   66 CONTINUE
C
      II = -3
      WRITE (6,201)   N
C
      DC 33  I = 1,65
         II = II+1
         IF  ( I .EQ. 65 )   YAXIS = YAXIS - 2*YDELTA
```

```
          IF   ( MOD(I,5) .NE. 3 )    GO TC 15
          YAXIS = SCORES(N) - YDELTA*II
          WRITE (6,200)    YAXIS,    ( GRAPH(I,J), J=1,112 )
          GO TO 33
   15     WRITE (6,400)    ( GRAPH(I,J), J=1,112 )
          IF   ( I .EQ. 65 )    WRITE (6,202)    YAXIS
   33 CONTINUE
C
      XAXIS(1) = X(1)
      DO 44   J = 2,12
   44     XAXIS(J) = XAXIS(J-1) + 10*XDELTA
C
      IF   ( X(N) .LT. .001   .OR.   X(N) .GE. 1.E04   )   GO TC 25
      NDIG =  INT ( 1. + ABS (ALOG10(X(N)) )   )
      NFRCT =   INT ( 1.5+ ABS (ALOG10 ( XRANGE/11. ) )   )
      IF   ( XRANGE .GE. 11. )    NFRCT = 1
      NDNF = NDIG+NFRCT
      IF   ( X(N) .LE. 1 )    NDNF = NFRCT
      IF   ( NFRCT .GT. 4   .OR.   NDNF .GE. 7 )    GC TO 25
      FMTF(2) = DIGITS(NFRCT+3)
      FMTF(11) = DIGITS(NFRCT)
      WRITE (6,FMTF)    ( XAXIS(J), J=1,11 )
      GO TO 40
   25 WRITE (6,600)    ( XAXIS(J), J=1,11,2 )
   40 WRITE (6,604)  XDELTA, YDELTA
C
      RETURN
  200 FCRMAT   ( 1X, F6.3, 1X, 112A1 )
  201 FORMAT   ( '1' ,   50X,   'NUMBER OF CRDERFD PAIRS   = ', I6 // 2X,
     *          'EXPCN. SCORES ' //)
  202 FCRMAT   (   '+', F6.3 )
  400 FORMAT   (   8X, 112A1  )
  600 FORMAT   ( 4X, 6(1PE11.4, 9X ) )
  604 FCRMAT   (////   8X, 'X-SCALE : ''*''   =  ',E10.3, '    UNITS',
     *           //  8X, 'Y-SCALE : ''*''   =   ', E1C.3 , '    UNITS' )
      END
```

```
C-------------------------------------------------------------------
C
C
C     SUBROUTINE  LIST
C
C
C     PURPOSE
C
C         SUBROUTINE  LIST  IS INDENDED TO :
C
C     1. ORDER A SET OF DATA IN ASCENDING ORDER.
C
C     2. PRINT THE SERIAL NUMBER OF DATA, THE VALUE OF ORDERED DATA
C        POINTS, THE FREQUENCY AND PERCENT OF EACH DATA POINT, AND
C        THE GRAPH OF PROBABILITY OF EACH DATA POINT.
C
C
C     CALLING SEQUENSES
C
C         CALL   LIST  ( X, N ).
C
C
C     ARGUMENTS
C
C         X    A SINGLE DIMENSIONAL ARRAY OF DATA (REAL *4).
C         N    NUMBER OF DATA POINTS.
C
C
C     USAGE
C
C         THE DATA WILL BE SORTED INTO INCREASING ORDER AND THEN
C     A FIVE-COLUMN OUTPUT WILL BE GIVEN.
C
C     THE FIRST COLUMN WILL GIVE THE SERIAL NUMBER OF ORDERED DATA
C     THE SECOND COLUMN WILL GIVE THE DATA POINT VALUES.
C     THE THIRD COLUMN WILL GIVE THE FREQUENCY OF OCCURENCE
C     OF EACH DATA POINT VALUE.
C     THE FORTH COLUMN WILL GIVE THE PERCENTAGE OF EACH DATA POINT VALUE
C     THE FIFTH COLUMN WILL GIVE A GRAPHICAL REPRESENTATION OF VALUES
C     OF THE FORTH COLUMN.
C
C
C     SUBROUTINES REQUIRED
C
C         SUBROUTINE 'PXSORT' IS USED TO ORDER THE DATA.
C     'PXSORT' HAS BEEN COMPILED AND ADDED TO MPSLIB.
C
C
```

```
C     PROGRAMMER :  GEORGIOS J. DANIKAS
C
C                   UNDER DIRECTICNS CF PROF.  P. A. W. LEWIS
C
C
C     DATE :  APR 77
C
C
C-------------------------------------------------------------------------
C
C
C
      SUBROUTINE   LIST  ( X, N )
C
C
      INTEGER  COUNT, SERNUM
      LOGICAL *1  STAR
      DATA    STAR  / '*' /
      DIMENSICN   X(N)
C
      SERNUM = 1
      CALL    PXSORT   ( X, 1, N )
C
C     FIND MAXIMUM FREQUENCY TO SCALE THE GRAPH PART
C
      I1 = 1
      PROMAX = 0.
   20 COUNT = 0
      TCP = X(I1)
C
      DC 22  I = I1,N
         K = I
         IF  ( X(I) .GT. TOP )   GO TC 10
         COUNT = COUNT + 1
   22 CONTINUE
C
   10 PROB = FLOAT(COUNT) / N
      PROMAX = AMAX1 ( PROMAX,PROB )
      I1 = K
      IF  ( K .EQ. N )   GO TO 25
      GO TO 20
C
C     FIND AND PRINT SERIAL NUMBER, FRECUENSIES, FRCBABILITIES
C     AND GRAPH THE PROBABILITY
C
   25 PROMIN = 1./N
      IF  ( PROMAX .LE. PROMIN )   GO TC 35
      TOP = X(1)
```

```
      I1 = 1
      WRITE (6,200)
    5 COUNT = 0
C
      DC 11  I = I1,N
         IF  ( X(I) .NE. TOP )  GO TO 15
         COUNT = COUNT + 1
   11 CONTINUE
C
   15 PROB = FLOAT (COUNT) / N
      NSTAR =  INT ( 40*PROB/PROMAX + .5 )
      IF  ( NSTAR .EQ. 0 )
     *WRITE (6,401)  SERNUM, X(SERNUM), COUNT, PRCB
      IF  ( NSTAR .GT. 0 )
     *WRITE (6,400)   SERNUM, X(SERNUM), COUNT, PRCB, ( STAR, J=1,NSTAR)
      SERNUM = SERNUM + COUNT
      IF  ( SFRNUM .EQ. (N+1) )   RETURN
      I1 = I
      TOP = X(I)
      GO TO 5
C
   35 WRITE (6,600)   ( X(I), I=1,N )
      RETURN
C
  200 FORMAT  ( '1',  5X, 'SERIAL NUMBER',  6X,'ORCERED DATA', 6X,
     *          'FREQUENCIES',6X,'PERCENT',18X,'PRCBABILITY  GRAPH'/// )
  400 FORMAT  ( 7X,I6, 9X,  E16.8,5X,I6,11X, F7.5,8X, 45A1 /  )
  401 FORMAT  ( 7X,I6, 9X,  E16.8,5X,I6,11X, F7.5 )
  600 FCRMAT  ( '1', 5X, 'EACH OF DATA POINTS HAS ITS OWN VALUE',
     *          //, 5X,' THE ORDERED CATA ARE', ///, 5( 5X, E15.7 )  )
      END
```

```
C-----------------------------------------------------------------------
C
C
C       SUBROUTINE  SECTN
C
C
C       PURPOSE
C
C           SUBROUTINE  SECTN  IS INTENDED TO :
C
C       1. SECTION A SET OF DATA INTO SEVERAL DISJOINT SECTIONS.
C       2. COMPUTE A SET OF BASIC STATISTICS FOR EACH SECTION.
C       3. COMPUTE A SET OF BASIC STATISTICS FOR THE ENTIRE SET OF DATA.
C       4. COMPUTE AN ESTIMATE OF EACH OF THE BASIC STATISTICS BY AVERAGE
C          THE SET OF BASIC STATISTICS  OBTAINED FROM THE SECTIONS.
C
C
C       CALLING SEQUENCES
C
C          CALL   SECTN  ( X, N, K )
C
C
C       ARGUMENTS
C
C           X    A SINGLE DIMENSION ARRAY (REAL*4) OF DATA.
C
C           N    NUMBER OF DATA.
C
C           K    NUMBER OF SECTIONS. ( MUST BE NO GREATER THAN 100 ).
C
C
C       USAGE
C
C          THE DATA WILL BE SECTIONED INTO  K SECTIONS AND FOR EACH
C       SECTION THE FOLLOWING STATISTICS WILL BE COMPUTED, USING THE
C       FORMULAS AS THEY ARE DESCRIBED IN 'SCR STAND. MATH. TABLES'
C       EDITION 22 (1974).
C
C          MEAN      AVERAGE OF EACH SECTION  ( P. 570 )
C
C          MEDIAN    MID-VALUE OF EACH SECTION IF N/K IS ODD
C                    AVERAGE OF THE TWO MID-VALUES OF EACH SECTION O.W.
C                    ( P. 571 )
C
C          VARIANCE THE UNBIASED ESTIMATE HAS BEEN USED.
C
C          STD. DEV.  TEH UNBIASED ESTIMATE HAS BEEN USED. ( P.573 )
C
```

```
C        COEF. OF VARIANCE = STAD. DEV./ |MEAN|
C
C        SKEWNESS  = M3 /  STAD. DEV.**3
C                    WHERE  M3 IS THE THIRD CENTRAL MOMENT
C                    UNBIASLLY COMPUTED
C
C        KURTOSSIS = M4 / VARIANCE** -3
C                    WHERE M4 IS THE FORTH CENTRAL MOMENT.
C
C        MINIMUM   = X(1)
C
C        MAXIMUM   = X(N)
C
C        THE SAME AS ABOVE  STATISTICS ARE COMPUTED FOR THE ENTIRE SET
C     OF DATA (USING THE SAME FORMULAS)
C
C        EVENTUALLY SOME ESTIMATES OF THE ABOVE STATISTICS ARE COMPUTED
C     USING THE RESULTS FROM EACH SECTION
C
C        NOTE : K MUST BE SUCH A NUMBER AS TO MINIMIZE THE NUMBER OF
C     DATA POINTS THAT WILL HAVE TO BE DISCARDED. SECTN PLACES THE DATA
C     INTO THE EQUAL SIZE SECTION DISCARDING ANY DATA LEFT OVER.
C
C        FOR K<=3 OR K>100 OR (N/K)<=3 ESTIMATES FROM UNSECTIONED DATA
C     WILL BE ONLY GIVEN AND NO ESTIMATES FOR THE ESTIMATED STATISTICS
C     WILL BE COMPUTED.
C
C        NO OUTPUT IS EXPECTED IF N<=3 .
C
C
C     SUBROUTINE REQUIREMENTS
C
C        SUBROUTINE 'PXSORT' IS USED AND IT HAS BEEN COMPILED  AND ADDED
C     TO MPSLIB
C
C
C     PROGRAMMER :  GEORGIOS J. DANIKAS
C
C                   UNDER DIRECTIONS OF PROF.  P. A. W. LEWIS
C
C     DATE       :  MAY 77
C
C-----------------------------------------------------------------------
C
C
```

```
      SUBROUTINE   SECTN  ( X, ND, K )
C
      DIMENSION   X(N), SORT(100)
      LOGICAL *1  PARMTR(70)
      REAL  *8    DIF, XM3, XM4, SUM1, SUM2, SUM3, SUM4
      COMMON   STAT(108,9)
      DATA   PARMTR  /   'M', 'E', 'A', 'N', ' ', ' ', ' ', ' ', ' ',' ',
     *                   'M','E','D','I','A','N',' ',' ',' ',' ',
     *                   'V', 'A', 'R', 'I', 'A', 'N', 'C', 'E', ' ',' ',
     *                   'S', 'T', 'D', '.', ' ', 'D', 'E', 'V', '.',' ',
     *                   'C', 'O', 'E', 'F', '.', ' ', 'V', 'A', 'R','.',
     *                   'S', 'K', 'E', 'W', 'N', 'E', 'S', 'S', ' ',' ',
     *                   'K', 'U', 'R', 'T', 'O', 'S', 'S', 'I','S',' ' /
      IF   ( ND .LE. 3 )   GO TO 10
      K1 = K+1
      IF   ( K .GT. 100 )   K1 = 101
      XK = K
      M = ND/K
      N = M*K
      IF   ( M .LE. 3 )   N = ND
C
C     COMPUTE DESIRED STATISTICS OF UNSECTIONED DATA
      CALL   ESTIMA  ( X, N, 0, N, K1 )
C
C     SECTION THE DATA AND ESTIMATE DESIRED STATISTICS FOR EACH SECTION.
      IF   ( K .GT. 100 )   GO TO 15
      IF   ( M .LE. 3 )   GO TO 15
      MODM2 = MOD(M,2)
      MODN2 = MOD(N,2)
      MODK2 = MOD(K,2)
      MO2 = M/2
      NO2 = N/2
      KO2 = K/2
      DO 11   J = 1,K
         L1 = (J-1)*M
         L2 = J*M
         CALL   ESTIMA  ( X, N, L1, L2, J )
         LP1 = L1+1
         CALL   PXSORT  ( X, LP1, L2 )
   11    STAT(J,2) = ((1 - MODM2 )*X(L1+MO2)+X(LP1+MO2) ) / (2 - MODM2 )
      CALL   PXSORT  ( X, 1, N )
      STAT(K1,2) = ( (1 - MODN2 )*X(NO2) + X(NO2+1) ) / ( 2 - MODN2  )
      IF   ( K .LE. 3 )   GO TO 30
C
C     COMPUTE DESIRED  STATISTICS FOR EACH ESTIMATED STATISTIC
      DO 22   J = 1,7
         DO 33   I = 1,K
   33       SORT(I) = STAT(I,J)
```

```
       K1J = K1+J
       CALL   ESTIMA   ( SORT, K, 0, K, K1J )
       CALL   PXSORT   ( SORT, 1, K )
       STAT(K1J,2) = ( (1 - MODK2 )*SQRT(KO2) + SQRT(KO2+1) )
   *                 / ( 2 - MODK2  )
22     STAT(K1J,8) = STAT(K1J,4) / SQRT(XK)
    WRITE (6,200)
    WRITE (6,400)    ( I, (STAT(I,J), J=1,9), I=1,K  )
    WRITE (6,600)    (STAT(K1,J), J = 1,9   )
    DO 44  J = 1,7
       JP =  (J-1)*10+1
       J10 = J*10
       K1J = K1+J
44     WRITE (6,800)    ( PARMTR(I), I=JP,J10 ),  (STAT(K1J,I), I=1,8)
    WRITE (6,801)
    RETURN
10  WRITE (6,201)
    RETURN
30  WRITE (6,200)
    CALL    PXSORT   ( X, 1, N )
    STAT(K1,2) = (  (1 - MODN2  )*X(NO2) + X(NO2+1) ) / ( 2 - MODN2   )
    WRITE (6,400)     ( I,  (STAT(I,J), J=1,9), I=1,K   )
    WRITE (6,604)     (STAT(K1,J), J = 1,9   )
    RETURN
15  WRITE (6,200)
    CALL    PXSORT   ( X, 1, N )
    STAT(K1,2) = (  (1 - MODN2  )*X(NO2) + X(NO2+1) ) / ( 2 - MODN2   )
    IF  ( M .LE. 3 )    WRITE (6,601)
    IF  ( K .GT. 100 )    WRITE (6,603)
    WRITE (6,602)   (   STAT(K1,J), J=1,9   )
    RETURN
200 FORMAT  ( '1'// 48X, 'ESTIMATED SAMPLE PARAMETERS',
   *          ////  3X, 'SECTION', 7X, 'MEAN', 7X, 'MEDIAN', 5X,
   *          'VARIANCE', 3X, 'STD. DEV.', 4X,'COEF VAR',4X,'SKEWNESS',
   *          3X, 'KURTOSSIS', 4X, 'MINIMUM', 5X, 'MAXIMUM' /   )
201 FORMAT  ( 5X, '*** SAMPLE SIZE TOO SMALL ***' )
400 FORMAT  ( / 4X, I3, 5X, 1P9E12.4 )
600 FORMAT  ( //1X, 'UNSECTIONED', 1P9E12.4 ,
   *          ////// 41X, 'ESTIMATED PARAMETERS OF SAMPLE PARAMETERS'
   *          ////  1X, 'PARAMETER', 7X, 'MEAN', 8X, 'MEDIAN', 6X,
   *          'VARIANCE', 4X, 'STD. DEV.',5X, 'COEF VAR', 5X,
   *          'SKEWNESS', 4X, 'KURTOSSIS', 2X, 'STD. DEV./NS**.5' / )
601 FORMAT  (/46X, '*** SECTION SIZE TOO SMALL ***'// 35X,
   *          'PARAMETERS FROM UNSECTIONED DATA HAVE BEEN ESTIMATED' )
602 FORMAT  ( //1X, 'UNSECTIONED', 1P9E12.4 )
603 FORMAT  ( /, 4X, '*** NUMBER OF SECTIONS GREATER THAN 100 .    ',
   *          'NO ESTIMATES BY SECTIONING CAN BE COMPUTED ***',///35X,
   *          'PARAMETERS FROM UNSECTIONED DATA HAVE BEEN ESTIMATED' )
```

```
604 FORMAT  ( //1X, 'UNSECTIONED', 1P9E12.4,///1X, '*** NO ESTIMATES',
   *          ' OF ESTIMATED STATISTICS CAN BE COMPUTED.  NUMBER OF ',
   *          'SECTIONS TOO SMALL ***' )
800 FORMAT  ( / 1X, 10A1, 1P7E13.4, 2X, 1PE12.4 )
801 FORMAT  ( ///104X, 'NS=# OF SECTS'  )
    END


    SUBROUTINE  ESTIMA  ( X, N, L1, L2, IS )


    DIMENSION  X(N)
    REAL *8   DIF, XM3, XM4, SUM1, SUM2, SUM3, SUM4
    COMMON       STAT(108,9)
    SUM1 = 0.D0
    M = L2-L1
    XN = M
    DO 11   I = 1,M
 11    SUM1  = SUM1+X(L1+I)
    STAT(IS,1) = SUM1/XN
    STAT(IS,8) = X(L1+1)
    STAT(IS,9) = X(L2)
    SUM2 = 0.D0
    SUM3 = 0.D0
    SUM4 = 0.D0
C
    DO 22   I = 1,M
       DIF = X(L1+I) - STAT(IS,1)
       SUM2 = SUM2 + DIF**2
       SUM3 = SUM3 + DIF**3
       SUM4 = SUM4 + DIF**4
       STAT(IS,8) = AMIN1 ( X(L1+I), STAT(IS,8) )
 22    STAT(IS,9) = AMAX1 ( X(L1+I), STAT(IS,9) )
    STAT(IS,3) = SUM2 / (XN-1)
    STAT(IS,4) = SQRT ( STAT(IS,3) )
    STAT(IS,5) = 0.
    IF  ( ABS (STAT(IS,1)) .GT. 1.E-30  )
   *      STAT(IS,5) = STAT(IS,4) / ABS (STAT(IS,1) )
    XM3 = XN*SUM3 / ( (XN-1)*(XN-2) )
    XM4 = SUM4* ( 3+XN*(XN-2) ) / ( (XN-1)*(XN-2)*(XN-3) )
   *      - 3*STAT(IS,3)**2*(XN-1)*(2*XN-3) / ( XN*(XN-2)*(XN-3) )
    STAT(IS,6) = XM3 / STAT(IS,4)**3
    STAT(IS,7) = XM4 / STAT(IS,3)**2-3.
    RETURN
    END
```

```
C-----------------------------------------------------------------------
C
C
C     SUBROUTINE  JACK
C
C
C     PURPOSE
C
C         SUBROUTINE  JACK  IS INTENDED TO :
C
C     1. GROUP A SET OF DATA INTO SEVERAL GROUPS
C     2. COMPUTE A SET OF PSEUDO-VALUES FOR JACKKNIFE ESTIMATES
C     3. COMPUTE A SET OF BASIC STATISTICS FOR THE ENTIRE SET OF DATA.
C     4. COMPUTE THE BASIC JACKKNIFE ESTIMATES
C     5. GIVE ESTIMATES OF THE VARIANCES OF THE JACKNIFED ESTIMATES.
C
C
C     CALLING SEQUENCES
C
C        CALL   JACK   ( X, XS, STAT, N, IG )
C
C
C     ARGUMENTS
C
C         X     A SINGLE DIMENSION ARRAY (REAL*4) OF DATA.
C
C        XS     A SINGLE DIMENSION ARRAY (REAL*4). RETURNS ORDERED DATA.
C
C       STAT    TWO DIMENSION DUMMY ARRAY (REAL*4). DIMENSIONED BY (IG,7)
C
C         N     NUMBER OF DATA.
C
C        IG     NUMBER OF GROUPS PLUS ONE (=R+1).  NO LESS THAN 3.
C
C
C     USAGE
C
C        THE DATA WILL BE GROUPED INTO R=IG-1 GROUPS AND FOR EACH SET OF
C     THE  (R-ITH) GROUP,I=1,2,...,R THE PSEUDO-VALUES FOR THE FOLLOWING
C     STATISTICS WILL BE COMPUTED  USING THE FORMULAS AS THEY ARE
C     DESCRIBED IN 'SCR STAND. MATH. TABLES', EDITION 22 (1974).
C
C        MEAN      AVERAGE OF EACH SECTION  ( P. 570 )
C
C        MEDIAN    MID-VALUE OF EACH SECTION IF N/K IS ODD
C                  AVERAGE OF THE TWO MID-VALUES OF EACH SECTION O.W.
C                  ( P. 571 )
C
```

```
C        VARIANCE THE UNBIASED ESTIMATE HAS BEEN USED.
C
C        STD. DEV.  TEH UNBIASED ESTIMATE HAS BEEN USED. ( P.573 )
C
C        COEF. OF VARIANCE = STAD. DEV./ |MEAN|
C
C        SKEWNESS  = M3 /  STAD. DEV.**3
C                    WHERE  M3 IS THE THIRD CENTRAL MOMENT
C                    UNBIASLLY COMPUTED
C
C        KURTOSSIS = M4 / VARIANCE** -3
C                    WHERE M4 IS THE FORTH CENTRAL MOMENT.
C
C        THE SAME AS ABOVE  STATISTICS ARE COMPUTED FOR THE ENTIRE SET
C     OF DATA (USING THE SAME FORMULAS)
C
C        USING THE COMPUTED AS ABOVE PSEUDO-VALUES, JACK ESTIMATES THE
C     BASIC (MEAN, MEDIAN, VARIANCE, STAD. DEV.,SKEWNESS, KURTOSSIS)
C     STATISTICS OF DATA BY JACKKNIFE METHOD AND THE VARIANCE AND STAND.
C     DEVIATION FOR EACH STATISTIC IS ALSO GIVEN.
C
C
C        NOTE : R=IG-1 MUST BE SUCH A NUMBER AS TO MINIMIZE THE NUMBER
C     OF DATA POINTS THAT WILL HAVE TO BE DISCARDED. JACK PLACES  THE
C     DATA INTO THE EQUAL SIZE GROUP DISCARDING ANY DATA LEFT OVER.
C
C
C     ERROR CONDITIONS
C
C        NO OUTPUT IS EXPECTED IF N<=3
C
C        IF IG<=2 OR (IG-2)*(N/(IG-1))<=3 ESTIMATES OF UNGROUPED DATA
C     WILL BE ONLY GIVEN.
C
C     SUBROUTINE REQUIREMENTS
C
C        SUBROUTINE 'PXSORT' IS USED AND IT HAS BEEN COMPILED  AND ADDED
C     TO MPSLIB
C
C
C     PROGRAMMER :  GEORIOS J. DANIKAS
C                   UNDER DIRECTIONS OF PROF.  P. A. W. LEWIS
C
C     DATE       :  MAY 77
C
C-----------------------------------------------------------------------
C
C
```

```
      SUBROUTINE   JACK  ( X, XS, STAT, ND, IG )

      DIMENSION  X(ND), XS(ND), STAT(IG,7)
      LOGICAL *1  PARMTR(70)
      DATA   PARMTR  /  'M', 'E', 'A', 'N', ' ', ' ', ' ', ' ', ' ',' ',
     *                  'M','E','D','I','A','N',' ',' ',' ',' ',
     *                  'V', 'A', 'R', 'I', 'A', 'N', 'C', 'E', ' ',' ',
     *                  'S', 'T', 'D', '.', ' ', 'D', 'E', 'V', '.',' ',
     *                  'C', 'O', 'E', 'F', '.', ' ', 'V', 'A', 'R','.',
     *                  'S', 'K', 'E', 'W', 'N', 'E', 'S', 'S', ' ',' ',
     *                  'K', 'U', 'R', 'T', 'O', 'S', 'S', 'I','S',' ' /
      IF  ( ND .LE. 3 )   GO TO 15
      IR = IG-1
      IF  ( IR .LT. 1 )   IR = 1
      NG = ND/IR
      N = NG*IR
C
C     COMPUTE ESTIMATES OF UNGROUPED DATA
C
      CALL   JACKES  ( X, STAT, IG, IG, N )
C
C     COMPUTE ESTIMATES OF EACH GROUP
C
      L2 = 5
      NO2 = N/2
      MODN2 = MOD(N,2)
      IF  ( IR .LT. 2 )   GO TO 25
      L2 = (IR-1)*NG
      IF  ( L2 .LE. 3 )   GO TO 25
      LO2 = L2/2
      MODL2 = MOD(L2,2)
      DO 11  I = 1,IR
         L1 = (I-1)*NG
         DO 22  J = 1,L2
            INDEX = J
            IF  ( J .GT. L1 )   INDEX = J+NG
   22       XS(J) = X(INDEX)
         CALL   JACKES  ( XS, STAT, I, IG, L2 )
         CALL   PXSORT  ( XS, 1, L2)
   11    STAT(I,2) = ( (1-MODL2)*XS(LO2) + XS(LO2+1) ) / ( 2-MODL2 )
      DO 33  I = 1,N
   33    XS(I) = X(I)
      CALL   PXSORT  ( XS, 1, N )
      STAT(IG,2) = ( (1-MODN2)*XS(NO2) + XS(NO2+1) ) / (2-MODN2)
C
      WRITE (6,200)
```

```
      WRITE (6,400)   ( I, (STAT(I,J), J=1,7), I=1,IR )
      WRITE (6,401)   ( STAT(IG,J), J=1,7 )
C
C     CCMPUTE JACKKNIFE ESTIMATES.
C

      DC 44  J = 1,7
         STAT(IG,J) = STAT(IG,J)*IR
         T1 = 0.
         PS1 = 0.
         PS2 = 0.
         DO 66  I = 1,IR
            T1 = T1+STAT(I,J)
            PS2 = PS2+(STAT(IG,J)-(IR-1)*STAT(I,J))**2
   66       PS1 =PS1+STAT(IG,J)-(IR-1)*STAT(I,J)
         STAT(1,J) = STAT(IG,J)-(IR-1)*T1/IR
         STAT(2,J) = (PS2-PS1**2/IR) / (IR-1)
   44    STAT(3,J) = SQRT (STAT(2,J)/IR)
      DC 77  J = 1,7
         JP = (J-1)*10+1
         J10 = J*10
   77    WRITE (6,600)    ( PARMTR(I), I=JP,J10 ),  (STAT(I,J), I=1,3 )
      RETURN
   15 WRITE (6,800)
      RETURN
   25 WRITE (6,200)
      IF  ( L2 .LE. 3 )  WRITE  (6,801)
      IF  ( IR .LT. 2 )  WRITE  (6,803)
      DC 88  I = 1,N
   88    XS(I) = X(I)
      CALL   PXSORT  ( XS, 1, N )
      STAT(IG,2) = ( (1-MODN2)*XS(NO2) + XS(NO2+1) ) / ( 2-MODN2 )
      WRITE (6,802)   ( STAT(IG,J), J=1,7 )
      RETURN
  200 FORMAT  ( '1'// 46X, 'ESTIMATED PARAMETERS',
     *          ////  3X, 'GROUP',10X,'MEAN',10X, 'MEDIAN',
     *          8X, 'VARIANCE',7X, 'STD. DEV.', 6X, 'COEF. VAR', 6X,
     *          'SKEWNESS', 7X, 'KURTCSSIS' // )
  400 FCRMAT  ( 3X, I3, 5X, 1P7E15.5 /)
  401 FORMAT  ( / 1X, 'UNGROUPED', 1X, 1P7E15.5 ,
     *        ////// 17X, 'ESTIMATED JACKKNIFE PARAMETERS'
     *        //// 1X, 'PARAMETER', 7X, 'JACKKNIFE', 7X, 'VARIANCE', 7X,
     *          'STD. DEV. OF ' / 46X, 'JACK. PARAMETER' / )
  600 FORMAT  ( / 1X, 10A1,  1P3E16.5 )
  800 FORMAT  ( //44X,'*** SAMPLE SIZE TOO SMALL ***' )
  801 FCRMAT  ( /43X, '*** INVALID GROUP SIZE ***' //,
     *     32X,'PARAMETERS FROM  UNGROUPED  DATA HAVE BEEN ESTIMATED'/)
  802 FORMAT  ( / 1X, 'UNGROUPED', 1X, 1P7E15.5,
```

```
     *           ////, 1X, 'JACKKNIFE PARAMETERS CANNOT BE ESTIMATED' )
  803 FORMAT   ( / 41X, '*** NUMBER OF GROUPS TOO SMALL ***' // 32X,
     *           'PARAMETERS FROM  UNGROUPED  DATA HAVE BEEN ESTIMATED'/)
      RETURN
      END


      SUBROUTINE    JACKES  ( X, STAT, IS, IG, N )


      DIMENSION X(N), STAT(IG,7)
      REAL *8   DIF, XM3, XM4, SUM1, SUM2, SUM3, SUM4
C
      XN = N
      SUM1 = 0.D0
      DO 11   I = 1,N
   11    SUM1  =  SUM1+X(I)
      STAT(IS,1) = SUM1/XN
      SUM2 = 0.D0
      SUM3 = 0.D0
      SUM4 = 0.D0
C
      DO 22   I = 1,N
         DIF = X(I) - STAT(IS,1)
         SUM2 = SUM2 + DIF**2
         SUM3 = SUM3 + DIF**3
   22    SUM4 = SUM4 + DIF**4
C
      STAT(IS,3) = SUM2 / (XN-1)
      STAT(IS,4) = SQRT ( STAT(IS,3) )
      STAT(IS,5) = 0.
      IF  ( ABS (STAT(IS,1)) .GT. 1.E-30  )
     *     STAT(IS,5) = STAT(IS,4) / ABS (STAT(IS,1) )
      XM3 = XN*SUM3 / ( (XN-1)*(XN-2) )
      XM4 = SUM4* ( 3+XN*(XN-2) ) / ( (XN-1)*(XN-2)*(XN-3) )
     *      - 3*STAT(IS,3)**2*(XN-1)*(2*XN-3) / ( XN*(XN-2)*(XN-3) )
      STAT(IS,6) = XM3 / STAT(IS,4)**3
      STAT(IS,7) = XM4 / STAT(IS,3)**2-3.
      RETURN
      END
```

```
C      THE GENERAL EARMA(P,Q) PROGRAM.
C      THIS  PROGRAM  CAN BE USED  TO SIMULATE
C      ALL THE EARMA(P,Q) CASES.


       INTEGER   G, QS, QS1, QX, QX1, CPS, CPX, CPG
       DIMENSION    US1(10000), US(10000), UX1(10000), UX(10000),
      *             FS(10000), FS1(10000), FX(10000), FX1(10000),
      *             SUMBS(31), SUMBS1(31), SUMBX(31), SUMBX1(31),
      *             EXP1(10030), EXPSL(10030), EXPXL(10030),
      *             BS(30), BS1(30), BX(30), BX1(30),
      *             SUMX(4,500), SUMXM(4,500), D(4,500), DM(4,500),
      *             SUMS(4,500 ), SUMSM(4,500 ),
      *             W(4,500 ), WB(4,500 ), WM(4,500 ), WMB(4,500 )
       COMMON     N
       DATA       BS(1), BS1(1), BX(1), BX1(1), SUMBS(1), SUMBS1(1),
      *           SUMBX(1), SUMBX1(1)  /  8*1.0  /
C
       READ  (5,100)    IUS1, IUS, IUX1, IUX, IE1, IS1, IX1, IFS, IFS1,
      *                 IFX1, IFX
       READ  (5,300)    N, M, KS, KS1, KX, KX1, QS, QS1, QX, QX1, CPS,
      *                 CPX, CPG
       READ  (5,301)    RX, RS, RHOS, RHOS1, RHOX, RHOX1
C
       WRITE (6,200)    N, M, KS, KS1, KX, KX1, QS, QS1, QX, QX1,
      *                 RX, RS, RHOS, RHOS1, RHOX, RHOX1, CPS, CPX, CPG
       IF  ( KS .LT. 1 )    GO TO 5
       READ  (5,301)    ( BS(I), I=1,KS )
       WRITE (6,202)    ( BS(I), I=1,KS )
       CALL    BETAS   ( SUMBS, KS, BS )
     5 IF  ( KS1 .LT. 1 )    GO TO 25
       READ  (5,301)    ( BS1(I), I=1,KS1 )
       WRITE (6,202)    ( BS1(I), I=1,KS1 )
       CALL    BETAS   ( SUMBS1, KS1, BS1 )
    25 IF  ( KX .LT. 1 )    GO TO 35
       READ  (5,301)    ( BX(I), I=1,KX )
       WRITE (6,202)    ( BX(I), I=1,KX )
       CALL    BETAS   ( SUMBX, KX, BX )
    35 IF  ( KX1 .LT. 1 )    GO TO 45
       READ  (5,301)    ( BX1(I), I=1,KX1 )
       WRITE (6,202)    ( BX1(I), I=1,KX1 )
       CALL    BETAS   ( SUMBX1, KX1, BX1 )


    45 J = 0
       WRITE (6,201) J, IUS1, IUS, IUX1, IUX, IE1, IS1, IX1, IFS, IFS1,
      *                 IFX1, IFX
```

```
      K = MAXO ( KS, KS1, KX, KX1, QS, CS1, QX, QX1 )
      N1 = N/4
      N2 = N/2
      N3 = 3*N/4
      NK = N+K
C
      SL = 1./RS
      XL = 1./RX
      CS = RX/RS
      CX = 1./CS
C
   10 J = J+1
      CALL   EXPON   ( IE1, EXP1, NK )
      CALL   EXPON   ( IX1, EXPXL, NK )
      CALL   EXPON   ( IS1, EXPSL, NK )
      CALL   RANDOM   ( IUX, UX, N )
      CALL   RANDCM   ( IUX1, UX1, N )
      CALL   RANDCM   ( IUS, US, N )
      CALL   RANDCM   ( IUS1, US1, N )
C
      DO 11  I = 1,NK
         EXPSL(I) = EXPSL(I)*SL
   11    EXPXL(I) = EXPXL(I)*XL
C
      IF  ( CPG .EQ. 0 )   GO TO 65
      IF  ( QS .EQ. 1 )    CALL   AUTOR   ( FS, EXP1, SL, RHOS, IFS, 1 )
      IF  ( QX .EQ. 1 )    CALL   AUTOR   ( FX, FXF1, XL, RHCX, IFX, 1 )
      IF  ( QS1 .EQ. 1 )   CALL   AUTOR   ( FS1, EXPSL, 1., RHOS1,IFS1,0)
      IF  ( QX1 .EQ. 1 )   CALL   AUTOR   ( FX1, EXPXL, 1., RHCX1,IFX1,0)
      GO TO 50
C
   65 IF  ( QS1 .EQ. 1  .AND.  CPS .EQ. 0 )   CALL   AUTOR   ( FS1,
     *    EXPSL, 1., RHOS1, IFS1, 1 )
      IF  ( QS1 .EQ. 1  .AND.  CPS .EQ. 1 )   CALL   AUTOR   ( FS1,
     *    EXPXL, CS, RHOS1, IFS1, 1 )
      IF  ( QX1 .EQ. 1  .AND.  CPX .EQ. 0 )   CALL   AUTOR   ( FX1,
     *    EXPXL, 1., RHCX1, IFX1, 1 )
      IF  ( QX1 .EQ. 1  .AND.  CPX .EQ. 1 )   CALL   AUTOR   ( FX1,
     *    EXPSL, CX, RHCX1, IFX1, 1 )
C
   50 ARIV = 0.
      SERV = 0.
      ARIVM = 0.
      SERVM = 0.
      WAIT = 0.
      SNEW = 0.
      WAITM = 0.
      WBAR = 0.
```

```
      WMBAR = 0.
C
      DO 22  I = 1,N
         SOLD = SNEW
         WOLD = WAIT
         WMOLD = WAITM
         IES = KS1 + I
         IEX = KX1 + I
         VAL = 0.
         IF  ( CPG .EQ. 1 )   GO TO 75
C
         SNEW = BS1(1) * EXPSL(IES)
         IF  ( CPS .EQ. 0  .AND.  KS1 .GT. 0 )    CALL  EARMA ( QS1,
     *       KS1, I, 1., BS1, EXPSL, FS1, SUMBS1, VAL, 0, US1(I)  )
         IF  ( CPS .EQ. 1  .AND.  KS1 .GT. 0 )    CALL   EARMA  ( QS1,
     *        KS1, I, CS, BS1, EXPXL, FS1, SUMBS1, VAL, 1, US1(I) )
         SNEW = SNEW + VAL
         IF  ( QS1 .EQ. 1  .AND.  KS1 .EQ. 0 )   SNEW = FS1(I)
         XNEW = BX1(1) * EXPXL(IEX)
         VAL = 0.
         IF  ( CPX .EQ. 0  .AND.  KX1 .GT. 0 )    CALL  EARMA ( QX1,
     *       KX1, I, 1., BX1, EXPXL, FX1, SUMBX1, VAL, 0, UX1(I)  )
         IF  ( CPX .EQ. 1  .AND.  KX1 .GT. 0 )    CALL   EARMA  ( QX1,
     *        KX1, I, CX, BX1, EXPSL, FX1, SUMBX1, VAL, 1, UX1(I) )
         XNEW = XNEW + VAL
         IF  ( QX1 .EQ. 1  .AND.  KX1 .EQ. 0 )   XNEW = FX1(I)
         GO TO 70
C
   75    IF  ( KS1 .GT. 0 )    CALL  EARMA  ( QS1, KS1, I, 1., BS1,
     *       EXPSL, FS1, SUMBS1, VAL, 0, US1(I) )
         SNEW = BS1(1) * EXPSL(IES) + VAL
         IF  ( QS1 .EQ. 1  .AND.  KS1 .EQ. 0 )   SNEW = FS1(I)
         VAL = 0.
         IF  ( KS .GT. 0 )   CALL   EARMA  ( QS, KS, I, SL, BS, EXP1,
     *       FS, SUMBS, VAL, 1, US(I) )
         SNEW = SNEW * BS(1) + VAL
         IF  ( QS .EQ. 1  .AND.  KS .EQ. 0 )   SNEW = FS(I)
         VAL = 0.
         IF  ( KX1 .GT. 0 )    CALL  EARMA  ( QX1, KX1, I, 1., BX1,
     *       EXPXL, FX1, SUMBX1, VAL, 0, UX1(I) )
         XNEW = BX1(1) * EXPXL(IEX) + VAL
         IF  ( QX1 .EQ. 1  .AND.  KX1 .EQ. 0 )   XNEW = FX1(I)
         VAL = 0.
         IF  ( KX .GT. 0 )   CALL   EARMA  ( QX, KX, I, XL, BX, EXP1,
     *       FX, SUMBX, VAL, 1, UX(I) )
         XNEW = XNEW * BX(1) + VAL
         IF  ( QX .EQ. 1  .AND.  KX .EQ. 0 )   XNEW = FX(I)
C
```

```
7C     ARIV = ARIV + XNEW
       SERV = SERV + SNEW
       ARIVM = ARIVM + EXPXL(I)
       SERVM = SERVM + EXPSL(I)
       IF  ( I .EQ. 1 )   GO TO 22
       WAIT = WOLD + SCLD - XNEW
       IF  ( WAIT .LE. 0. )   WAIT = 0.
       WAITM = WMOLD + EXPSL(I-1) - EXPXL(I)
       IF  ( WAITM .LE. 0. )   WAITM = 0.
       WBAR = WBAR + WAIT
       WMBAR = WMBAR + WAITM
       IF  ( I .EQ. N1  .OR.  I .EQ. N2  .OR.  I .EQ. N3  .OR.
     *       I .EQ. N )   GO TO 85
       GO TO 22
C
85     G = I/N1
       SUMX(G,J) = ARIV
       SUMS(G,J) = SERV
       W(G,J) = WAIT
       WB(G,J) = WBAR/I
       SUMXM(G,J) = ARIVM
       SUMSM(G,J) = SERVM
       WM(G,J) = WAITM
       WMB(G,J) = WMBAR/I
       D(G,J) = (SERV - ARIV)/I - SL + XL
       DM(G,J) = (SERVM - ARIVM)/I -SL + XL
22  CONTINUE
C
    WRITE (6,201) J, IUS1, IUS, IUX1, IUX, IE1, IS1, IX1, IFS, IFS1,
     *                  IFX1, IFX
    IF  ( J .EQ. M )   GO TO 15

    GC TO 10


C   OUTPUT PART
C
15  WRITE (6,211)   ( SUMX (1,I),  I = 1,J )
    WRITE (6,212)   ( SUMX (2,I),  I = 1,J )
    WRITE (6,214)   ( SUMX (4,I),  I = 1,J )
    WRITE (6,222)   ( SUMS (2,I),  I = 1,J )
    WRITE (6,224)   ( SUMS (4,I),  I = 1,J )
    WRITE (6,231)   (  W  (1,I),  I = 1,J  )
    WRITE (6,232)   (  W  (2,I),  I = 1,J  )
    WRITE (6,233)   (  W  (3,I),  I = 1,J  )
    WRITE (6,234)   (  W  (4,I),  I = 1,J  )
    WRITE (6,241)   (  WB (1,I),  I = 1,J  )
```

```
WRITE (6,242)   (  WB (2,I),   I = 1,J   )
WRITE (6,243)   (  WB (3,I),   I = 1,J   )
WRITE (6,244)   (  WB (4,I),   I = 1,J   )
WRITE (6,256)   (    D(2,I),   I = 1,J   )
WRITE (6,258)   (    D(4,I),   I = 1,J   )
WRITE (6,412)   ( SUMXM(2,I),   I = 1,J  )
WRITE (6,414)   ( SUMXM(4,I),   I = 1,J  )
WRITE (6,422)   ( SUMSM(2,I),   I = 1,J  )
WRITE (6,424)   ( SUMSM(4,I),   I = 1,J  )
WRITE (6,431)   (  WM  (1,I),  I = 1,J   )
WRITE (6,432)   (  WM  (2,I),  I = 1,J   )
WRITE (6,433)   (  WM  (3,I),  I = 1,J   )
WRITE (6,434)   (  WM  (4,I),  I = 1,J   )
WRITE (6,441)   (  WMB(1,I),   I = 1,J   )
WRITE (6,442)   (  WMB(2,I),   I = 1,J   )
WRITE (6,443)   (  WMB(3,I),   I = 1,J   )
WRITE (6,444)   (  WMB(4,I),   I = 1,J   )
WRITE (6,456)   (    DM(2,I),  I = 1,J   )
WRITE (6,458)   (    DM(4,I),  I = 1,J   )
WRITE (7,250)   ( SUMX (2,I),   I = 1,J  )
WRITE (7,250)   ( SUMX (4,I),   I = 1,J  )
WRITE (7,250)   ( SUMS (2,I),   I = 1,J  )
WRITE (7,250)   ( SUMS (4,I),   I = 1,J  )
WRITE (7,260)   ( W   (1,I),   I = 1,J )
WRITE (7,260)   ( W   (2,I),   I = 1,J )
WRITE (7,260)   ( W   (3,I),   I = 1,J )
WRITE (7,260)   ( W   (4,I),   I = 1,J )
WRITE (7,260)   ( WB  (1,I),   I = 1,J )
WRITE (7,260)   ( WB  (2,I),   I = 1,J )
WRITE (7,260)   ( WB  (3,I),   I = 1,J )
WRITE (7,260)   ( WB  (4,I),   I = 1,J )
WRITE (7,270)   (     D (2,I),   I = 1,J )
WRITE (7,270)   (     D (4,I),   I = 1,J )
WRITE (7,250)   ( SUMXM(2,I),   I = 1,J  )
WRITE (7,250)   ( SUMXM(4,I),   I = 1,J  )
WRITE (7,250)   ( SUMSM(2,I),   I = 1,J  )
WRITE (7,250)   ( SUMSM(4,I),   I = 1,J  )
WRITE (7,260)   ( WM  (1,I),   I = 1,J )
WRITE (7,260)   ( WM  (2,I),   I = 1,J )
WRITE (7,260)   ( WM  (3,I),   I = 1,J )
WRITE (7,260)   ( WM  (4,I),   I = 1,J )
WRITE (7,260)   ( WMB(1,I),    I = 1,J )
WRITE (7,260)   ( WMB(2,I),    I = 1,J )
WRITE (7,260)   ( WMB(3,I),    I = 1,J )
WRITE (7,260)   ( WMB(4,I),    I = 1,J )
WRITE (7,270)   (    DM (2,I),   I = 1,J )
WRITE (7,270)   (    DM (4,I),   I = 1,J )
```

```
  250 FORMAT  ( 10F8.3 )
  260 FORMAT  ( 10F8.4 )
  270 FORMAT  ( 10F8.5 )
  211 FORMAT  ( '1', / 64X, 'SUM X1      ', //// ( 5X, 10F12.4 / ) )
  212 FORMAT  ( /////  64X, 'SUM X2      ', //// ( 5X, 10F12.4 / ) )
  214 FORMAT  ( ///// 64X,  'SUM X4      ', //// ( 5X, 10F12.4 / ) )
  222 FORMAT  ( ///// 64X,  'SUM S2      ', //// ( 5X, 10F12.4 / ) )
  224 FORMAT  ( ///// 64X,  'SUM S4      ', //// ( 5X, 10F12.4 / ) )
  231 FORMAT  ( '1', / 64X, 'WAIT  1     ', //// ( 5X, 10F12.4 / ) )
  232 FORMAT  ( /////  64X, 'WAIT  2     ', //// ( 5X, 10F12.4 / ) )
  233 FORMAT  ( /////  64X, 'WAIT  3     ', //// ( 5X, 10F12.4 / ) )
  234 FORMAT  ( /////  64X, 'WAIT  4     ', //// ( 5X, 10F12.4 / ) )
  241 FORMAT  ( '1', / 64X, 'WAIT  BAR 1', //// ( 5X, 10F12.4 / ) )
  242 FORMAT  ( /////  64X, 'WAIT  BAR 2', //// ( 5X, 10F12.4 / ) )
  243 FORMAT  ( /////  64X, 'WAIT  BAR 3', //// ( 5X, 10F12.4 / ) )
  244 FORMAT  ( /////  64X, 'WAIT  BAR 4', //// ( 5X, 10F12.4 / ) )
  256 FORMAT  ( /////  64X, '    D 2     ', //// ( 5X, 10F12.6 / ) )
  258 FORMAT  ( /////  64X, '    D 4     ', //// ( 5X, 10F12.6 / ) )
  412 FORMAT  ( ///// 64X,  'SUM XM2     ', //// ( 5X, 10F12.4 / ) )
  414 FORMAT  ( ///// 64X,  'SUM XM4     ', //// ( 5X, 10F12.4 / ) )
  422 FORMAT  ( ///// 64X,  'SUM SM2     ', //// ( 5X, 10F12.4 / ) )
  424 FORMAT  ( ///// 64X,  'SUM SM4     ', //// ( 5X, 10F12.4 / ) )
  431 FORMAT  ( '1', / 64X, 'WAIT M1     ', //// ( 5X, 10F12.4 / ) )
  432 FORMAT  ( /////  64X, 'WAIT M2     ', //// ( 5X, 10F12.4 / ) )
  433 FORMAT  ( /////  64X, 'WAIT M3     ', //// ( 5X, 10F12.4 / ) )
  434 FORMAT  ( /////  64X, 'WAIT M4     ', //// ( 5X, 10F12.4 / ) )
  441 FORMAT  ( '1', / 64X, 'WAITM BAR 1', //// ( 5X, 10F12.4 / ) )
  442 FORMAT  ( /////  64X, 'WAITM BAR 2', //// ( 5X, 10F12.4 / ) )
  443 FORMAT  ( /////  64X, 'WAITM BAR 3', //// ( 5X, 10F12.4 / ) )
  444 FORMAT  ( /////  64X, 'WAITM BAR 4', //// ( 5X, 10F12.4 / ) )
  456 FORMAT  ( /////  64X, '    DM 2    ', //// ( 5X, 10F12.6 / ) )
  458 FORMAT  ( /////  64X, '    DM 4    ', //// ( 5X, 10F12.6 / ) )
C
  100 FORMAT  ( 5I16 )
  200 FORMAT  ( '1',40X, 'N = ', I6, 10X, 'M = ', I4, ///, 5X,
     *        'KS = ', I3,5X, 'KS1 = ',I3,5X,'KX = ',I3,5X,'KX1 = ',I3,//,
     *        5X, 'QS = ', I2,5X, 'QS1 = ',I2,5X,'QX = ',I2,5X,'QX1 = ',I2
     *        ,//, 5X, 'RX = ', F5.2, 5X, 'RS = ', F5.2, //
     *       ,5X,'RHOS = ',F5.2,5X,'RHOS1 = ',F5.2,5X,'RHOX = ',F5.2,
     *        5X,'RHOX1 =',F5.2, 5X, 'CPLS = ', I2, 5X, 'CPLX = ', I2,
     *         5X, 'CPLG = ', I2, / )
  201 FORMAT  ( /,5X,'J =',I4, 5X, 'SEEDS : ', 6I15 / 25X, 5I15 )
  202 FORMAT  ( /, 5X, 12F10.3, / )
  300 FORMAT  ( 13I5 )
  301 FORMAT  ( 16F5.3 )
      STOP
      END
```

```
      SLBROUTINE   BFTAS  ( SUM, K, B )

      DIMENSION  SUM(31), B(30)
      SUM(1) = B(1)
      PROD = 1
      IF  ( K .EQ. 1 )   GO TO 15
      DO 11  I = 2,K
         PROD = PROD*(1.-B(I-1))
   11    SUM(I) = SUM(I-1) + PROD*B(I)
   15 SUM(K+1) = SUM(K) + PROD*( 1.-B(K) )
      RETURN
      END


      SLBROUTINE   AUTOR  ( F, EXP, L, R, ISD , I1 )


      REAL  L
      COMMON  N
      DIMENSION  F(10000),  EXP(10030),  U(10000)
      F(1) = L* (I1*R+1-I1) * EXP(1)
      CALL   RANDOM  ( ISD, U, N )
      IF  ( I1*U(1) .GT. R )   F(1) = F(1) + L*EXP(2)
      DO 11  I = 2,N
         F(I) = R*F(I-1)
         IF  ( U(I) .GT. R )   F(I) = F(I) + L*EXP(I+I1)
   11 CONTINUE
      RETURN
      END


      SLBROUTINE    EARMA  ( IQ, K, IS, L, B, EXP, F, SUM, VAL, I1, U )


      REAL   L
      DIMENSION  B(31), EXP(10030), F(1C000), SUM(31)
      VAL = 0.
      IF  (  U   .LE. SUM(1) )   RETURN
      KP1 = K+1
      IF  ( K .EQ. 1 )   GO TO 15
C
      DO 11  ID = 2,KP1
         IR = ID
         IF  (  U   .LE. SUM(IR) )   GC TO 10
   11 CONTINUE
```

```
1C IS1 = IS+I1+KP1
   IRM1 = IR
   IF  ( IR .EQ. KP1 )    IRM1 = IR-1
   DO 22  I = 2,IRM1
22     VAL = VAL + B(I)*L*EXP(IS1-I)
   IF  ( IR .EQ. KP1 )    VAL = VAL + L*(1-IQ)*EXP(IS+I1) + F(IS)*IQ
   RETURN
15 VAL = L*(1-IQ)*EXP(IS+I1) + F(IS)*IQ
   RETURN
   END
```

```
C     THIS PROGRAM CAN BE USED ONLY FOR THE PARTICULAR
C     EARMA(1,0) CASE. IT IS A MODIFICATION OF THE GENERAL
C     EARMA(P,Q) PROGRAM.

      REAL *8      ARIV, SERV, WBAR, ARIVM, SERVM, WMBAR
      INTEGER      G
      DIMENSION    U(10000), F(10000), EX(10000), ES(10001),
     *             SUMX(500), SUMS(500), SUMXM(50C), SUMSM(500),
     *             D(500), DM(500), W(500), WB(500), WM(500), WMB(5C0)
      CCMMON       N
      READ  (5,300)   N, RX, RS, RHOS
      WRITE (6,20C)   N, RX, RS, RHOS
      READ  (5,100)  G, K
      REAC  (5,100)  IF, IEX, IES
      J = 0
      WRITE (6,201)  J, IF, IEX, IES
      NK = N + K
      SL = 1./RS
      XL = 1./RX
C
   10 J = J + 1
      CALL   EXPON  ( IEX, EX, N )
      CALL   EXPON  ( IES, ES, NK )
C
      DC 11  I = 1,N
         ES(I) = ES(I) * SL
   11    EX(I) = EX(I) * XL
C
      READ  (5,304)  ARIV, SFRV, SNEW, WAIT, WBAR, ARIVM, SERVM,
     *               SMNEW, WAITM, WMBAR
      WRITE (6,304)  ARIV, SERV, SNEW, WAIT, WBAR, ARIVM, SERVM,
     *               SMNEW, WAITM, WMBAR
      CALL   AUTOR  ( F, ES, RHOS, IF, K, SNEW )
      WRITE (6,201)  J, IF, IEX, IES
C
      DC 22  I = 1,N
         SOLD = SNEW
         WOLD = WAIT
         WMOLD = WAITM
         SMOLD = SMNEW
         ARIV = ARIV + EX(I)
         SERV = SFRV + F(I)
         ARIVM = ARIVM + EX(I)
         SERVM = SERVM + ES(I)
         SMNEW = ES(I)
         SNEW = F(I)
         WAIT = WCLD + SOLD - EX(I)
```

```
            IF  ( WAIT .LE. 0. )   WAIT = 0.
            WAITM = WMOLD + SMOLD - EX(I)
            IF  ( WAITM .LE. 0.)    WAITM = 0.
            WBAR = WBAR + WAIT
            WMBAR = WMBAR + WAITM
   22 CONTINUE
C
      SUMX(J) = ARIV
      SUMS(J) = SERV
      W(J) = WAIT
      WB(J) = WBAR / N/G
      SUMXM(J) = ARIVM
      SUMSM(J) = SERVM
      WM(J) = WAITM
      WMB(J) = WMBAR / N/G
      D(J) = (SERV - ARIV) / N/G - SL + XL
      DM(J) = (SERVM - ARIVM) / N/G - SL + XL
      WRITE (6,205)  ARIV, SERV, SNEW, WAIT, WBAR, ARIVM, SERVM,
     *               SMNEW, WAITM, WMBAR
      WRITE (7,303)  J, ARIV, SERV, SNEW, WAIT, WBAR, ARIVM, SERVM,
     *               SMNEW, WAITM, WMBAR
      IF  ( J .EQ. 500)     GO TO 15
      GO TO 10
C
   15 WRITE (7,203)  IF, IEX, IES
  100 FORMAT   ( 3I20 )
  200 FORMAT   ( '1', 57X, 'N = ', I6, ///, 5X,
     *            'RX = ', F6.3, 10X, 'RS = ', F6.3,  //, 5X, 'RHOS = ',
     *            F6.3, / )
  201 FORMAT   ( /, 5X, 'J = ', I4, 5X, 'SEEDS : ', 5I20 )
  203 FORMAT   ( 3I20 )
  205 FORMAT   ( /, 26X, 'ARIV = ', F10.3, 5X, 'SERV = ', F10.3, 5X,
     *            'SNEW = ', F8.3, 5X, 'WAIT = ', F9.4, 5X, 'WBAR = ',F14.3
     *            , /, 26X, 'ARVM = ', F10.3, 5X, 'SRVM = ', F10.3, 5X,
     *            'SMNW = ', F8.3, 5X, 'WATM = ', F9.4, 5X, 'WMBR = ',
     *            F14.3, // )
  300 FORMAT   ( I10, 3F10.3 )
  301 FORMAT   ( 3F10.5 )
  303 FORMAT   ( I3,2X,5F15.4,/,5X, 5F15.4 )
  304 FORMAT   ( 5X, 5F15.4, /, 5X, 5F15.4 )
      WRITE (6,212)   (   SUMX(I),   I = 1,J )
      WRITE (6,222)   (   SUMS(I),   I = 1,J )
      WRITE (6,232)   (      W (I),   I = 1,J )
      WRITE (6,242)   (     WB(I),   I = 1,J )
      WRITE (6,256)   (      D (I),   I = 1,J )
      WRITE (6,412)   (  SUMXM(I),   I = 1,J )
      WRITE (6,422)   (  SUMSM(I),   I = 1,J )
      WRITE (6,432)   (     WM (I),   I = 1,J )
```

```
      WRITE (6,442)   (    WMB(I),   I = 1,J )
      WRITE (6,456)   (    DM (I),   I = 1,J )
      WRITE (7,250)   (   SUMX(I),   I = 1,J )
      WRITE (7,250)   (   SUMS(I),   I = 1,J )
      WRITE (7,260)   (      W (I),  I = 1,J )
      WRITE (7,260)   (      WB(I),  I = 1,J )
      WRITE (7,270)   (      D (I),  I = 1,J )
      WRITE (7,250)   (  SUMXM(I),   I = 1,J )
      WRITE (7,250)   (  SUMSM(I),   I = 1,J )
      WRITE (7,260)   (     WM (I),  I = 1,J )
      WRITE (7,260)   (     WMB(I),  I = 1,J )
      WRITE (7,270)   (     DM (I),  I = 1,J )
  250 FORMAT  ( 10F8.1 )
  260 FORMAT  ( 10F8.3 )
  270 FORMAT  ( 10F8.5 )
  212 FORMAT  ( /////  64X, 'SUM X2      ', //// ( 5X, 10F12.4 / ) )
  222 FORMAT  ( ///// 64X,  'SUM S2      ', //// ( 5X, 10F12.4 / ) )
  232 FORMAT  ( /////  64X, 'WAIT  2     ', //// ( 5X, 10F12.4 / ) )
  242 FORMAT  ( /////  64X, 'WAIT  BAR 2', //// ( 5X, 10F12.4 / ) )
  256 FORMAT  ( /////  64X, '    D 2     ', //// ( 5X, 10F12.6 / ) )
  412 FORMAT  ( ///// 64X,  'SUM XM2     ', //// ( 5X, 10F12.4 / ) )
  422 FORMAT  ( ///// 64X,  'SUM SM2     ', //// ( 5X, 10F12.4 / ) )
  432 FORMAT  ( /////  64X, 'WAIT M2     ', //// ( 5X, 10F12.4 / ) )
  442 FORMAT  ( /////  64X, 'WAITM BAR 2', //// ( 5X, 10F12.4 / ) )
  456 FORMAT  ( /////  64X, '    DM 2    ', //// ( 5X, 10F12.6 / ) )
      STOP
      END


      SUBROUTINE   AUTOR  ( F, EX, R, ID, K, S )


      COMMON        N
      DIMENSION     F(10000), EX(10001), U(10000)
      F(1) = R * EX(1) * K  +  (1-K) * S * R
      CALL   RANDOM  ( ID, U, N )
      IF  ( U(1) .GT. R )     F(1) = F(1) + EX(K+1)
C
      DO 11   I = 2,N
         F(I) = R * F(I-1)
         IF  ( U(I) .GT. R )    F(I) = F(I) + EX(I+K)
   11 CONTINUE
C
      RETURN
      END
```

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